Carleman approximation without critical points

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Abstract

We present the class of semi-admissible subsets of an open Riemann surface on which Carleman approximation by non-critical holomorphic functions is possible. Using this result, we can characterize closed sets with empty interior on which one can approximate continuous functions by non-critical holomorphic ones. We also show how to use this result to construct a non-critical entire function with arbitrary asymptotic values.

The classical Carleman approximation theorem states that for any complex valued continuous function $f\colon \mathbb{R} \to \mathbb{C}$ and any continuous positive valued function $\varepsilon\colon \mathbb{R} \to (0,\infty)$, there exists an entire function $F\colon \mathbb{C} \to \mathbb{C}$ such that $|F(x) - f(x)| < \varepsilon(x)$ holds for every $x \in \mathbb{R}$. The distinguishing feature of Carleman approximation is that ε is a function, thus the difference between the original function f and the approximating function F can be controlled pointwise. In particular, the difference can be made arbitrarily small as we go off to infinity along the real line. Thus, this is a stronger result than just *uniform approximation*, where ε would be a constant function.

This result was proven by T. Carleman in [3] and was later generalized to other subsets of the complex plane and even to subsets of open Riemann surfaces. What exactly generalized means in this context is made more precise by the next definition.

Definition 1. *We say that a closed subset E of an open Riemann surface X is a set of* Carleman approximation*, if for every continuous positive valued function $\varepsilon\colon E \to (0,\infty)$ and every function $f \in \mathcal{A}(E)$ there exists a global holomorphic function $F \in \mathcal{O}(X)$ such that $|F(p) - f(p)| < \varepsilon(p)$ holds for every $p \in E$.*

Contrast this with the weaker notion of uniform approximation. As already mentioned, in this case, the difference between the original and approximating function is bounded by a constant.

Definition 2. *We say that a closed subset E of an open Riemann surface X is a set of* uniform approximation*, if for every function $f \in \mathcal{A}(E)$ and $\varepsilon > 0$ there exists a global holomorphic function $F \in \mathcal{O}(X)$ such that $|F(p) - f(p)| < \varepsilon$ holds for every $p \in E$.*

Let us clear up some notation. For an open set $\Omega \subseteq X$ we denote by $\mathcal{O}(\Omega)$ the class of holomorphic functions on Ω. For a closed set $E \subseteq X$ let $\mathcal{C}(E)$ denote the class of continuous functions on E and $\mathcal{A}(E)$ the class of functions which are continuous on E and holomorphic in its interior, that is $\mathcal{A}(E) = \mathcal{C}(E) \cap \mathcal{O}(\mathring{E})$.

The question thus becomes which closed subsets of the open Riemann surface X are sets of Carleman approximation. A complete characterization of such sets was given by A. Boivin in [2]. In what follows X^* represents the one point compactification of the open Riemann surface X. In the case when $X = \mathbb{C}$, this is just the Riemann sphere, which we will denote by $\widehat{\mathbb{C}}$.

Theorem 1. *A closed set $E \subseteq X$ in an open Riemann surface X is a set of Carleman approximation if and only if it satisfies the following three properties:*

(i) *The set* $X^* \setminus E$ *is connected.*

(ii) *The set* $X^* \setminus E$ *is locally connected (at* ∞*).*

(iii) *For each compact set* $K \subseteq X$ *there exists a compact set* $K \subseteq Q \subseteq X$ *such that no component of the interior* $\mathrm{int}(E)$ *of* E *meets both* K *and* $X \setminus Q$*.*

Let us investigate these three conditions more precisely. Condition (i) has to do with the notion of holes. Recall, that for a closed set $E \subseteq X$ a *hole* is a relatively compact connected component of $X \setminus E$. We denote the union of all the holes of the set E by $h(E)$.

Consider for example the unit circle $\mathbb{S}^1$ in the complex plane. Its complement has two connected components, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and $\{z \in \mathbb{C} \mid |z| > 1\}$, but only the first is relatively compact. Thus $h(\mathbb{S}^1) = \mathbb{D}$.

It is an easy exercise in topology to show that a closed set E has no holes if and only if the set $X^* \setminus E$ is connected. Having no holes is quite important in complex approximation theory as it is a necessary condition to achieve even uniform approximation on compact sets. For compact sets, Mergelyan's theorem tells us that it is also a sufficient condition.

Now let's investigate condition (ii). Here we come upon the following notion.

Definition 3. *A closed set* $E \subseteq X$ *has the* bounded exhaustion hull property *or* BEH property*, if the set* $h(E \cup K)$ *is relatively compact for every compact set* $K \subseteq X$*.*

Let's look at some examples. The horizontal strip $E = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \leq 1\} \subseteq \mathbb{C}$ has the BEH property, since for any compact $K \subseteq \mathbb{C}$ the holes of $E \cup K$ are contained in some bounded region. It's also easy to see that every compact set has the BEH property. For a non-example consider E to be the closure of the graph of the function $f(x) = \frac{1}{x}\sin\left(\frac{1}{x}\right)$ on $(0, \infty)$ as a subset of $\mathbb{C}$. Taking K to be a closed disc around the origin, we see that the holes of $E \cup K$ stretch out to infinity, so the set $h(E \cup K)$ is not relatively compact.

A slightly more involved exercise in topology is to show that a set E has the BEH property if and only if $X^* \setminus E$ is locally connected at infinity. It turns out that the BEH property is a necessary condition for uniform approximation on general closed sets, as it ensures that the set behaves nicely near infinity. Thus, if we want to have uniform approximation on a closed set, the set must be without holes and have the BEH property. In the case when $X = \mathbb{C}$, such sets are called *Arakelyan sets* and the famous Arakelyan approximation theorem states that these two necessary conditions are also sufficient.

Finally let's consider condition (iii). The statement of this condition is a bit more cumbersome, so lets again look at some examples. Taking $E = \{z \in \mathbb{C} \mid |\mathrm{Im}(z)| \leq 1\}$ to be a strip and $K = \overline{\mathbb{D}}$, shows that a strip does not satisfy this condition, since the interior of the strip certainly hits both K and the complement of any compact Q containing K.

On the other hand, the set $E = \bigcup_{m\in\mathbb{Z}} \left\{z \in \mathbb{C} \mid |z - m| \leq \frac{1}{3}\right\}$, which is a union of closed disjoint discs, does satisfy condition (iii). For a given compact set $K \subseteq \mathbb{C}$ let Q be the union of K and all the discs of E that K intersects. An interior component of E is just the interior of one of the discs and is thus clearly contained in K or in $\mathbb{C} \setminus Q$.

Condition (iii) comes into play when we want to go beyond uniform approximation and want ε to be a function. In his paper [5], P. Gauthier showed the following.

Theorem 2. *A closed set* $E \subsetneq X$ *that does not satisfy condition (iii) is a* set of uniqueness *i.e. there exist a continuous function* $\lambda\colon X \to (0, \infty)$ *such that if a global holomorphic function* $F \in \mathcal{O}(X)$ *satisfies*

$$|F(p)| < \lambda(p)$$

for every $p \in E$ *then* $F \equiv 0$*.*

The idea is that a set of uniqueness gives rise to a positive valued continuous function λ which on the set E goes to zero so fast, that the only holomorphic function on E which is in modulus smaller than λ is the constant 0 function. Using the function λ, one can then show that a set of uniqueness can't also be a set of Carleman approximation. This shows that all three conditions from Boivin's theorem are indeed necessary.

Motivated by the work of F. Forstnerič in [4], we consider the problem of Carleman approximation by non-critical functions on open Riemann surfaces, namely under what assumptions can one require that the global approximating function has no critical points. Recall that $p \in X$ is a *critical point* of the holomorphic function $F \in \mathcal{O}(X)$, if the differential of F vanishes at p and F is *non-critical* if it has no critical points. In the case where $X = \mathbb{C}$, the function F is non-critical if its derivative is nowhere vanishing.

To solve this problem, we introduce the following class of subsets of an open Riemann surface X.

Definition 4. *We say that a closed set $E \subseteq X$ is* semi-admissible*, if there exists a locally finite pairwise disjoint family of compact sets $\{H_\lambda\}_{\lambda\in\Lambda}$ and a closed set S with empty interior, such that $E = S \cup H$, where $H = \bigcup_{\lambda\in\Lambda} H_\lambda$.*

To summarize, semi-admissible sets are made up of two parts; a closed set with empty interior and a union of compact sets which are nicely separated. This encodes the most important property of semi-admissible sets, namely, that there is some room between the parts of the set with non-empty interior. The name comes from the fact that semi-admissible sets are a generalization of admissible sets, which are a standard notion in the theory of complex approximation. It also turns out that semi-admissible sets always satisfy condition (iii) of Boivin's characterization, which quickly follows from the fact that the compact sets H_λ are locally finite and pairwise disjoint. There do, however, exist sets which satisfy condition (iii), but are not semi-admissible, consider Figure 1.

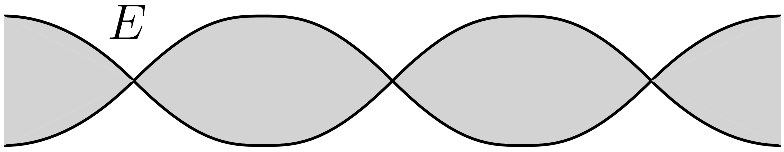

Figure 1: This set is not semi-admissible, but it does satisfy condition (iii).

We will also be focusing on a particular class of functions, which is defined as follows.

Definition 5. *Let $E \subseteq X$ be a semi-admissible set of the form $E = H \cup S$. Then $\widetilde{\mathcal{A}}(E)$ is the class of continuous functions on E which are holomorphic on some open neighborhood of the set H.*

The reason we consider functions in $\widetilde{\mathcal{A}}(E)$ instead of those in $\mathcal{A}(E)$ is that they give us better control over the critical points near the boundary, since they are defined on a neighborhood of the compact sets making up E.

A function $f \in \widetilde{\mathcal{A}}(E)$ is called *non-critical* if it has no critical points, where it is holomorphic. We are now able to state the main theorem.

Main Theorem. *Let $E \subseteq X$ be a semi-admissible set such that $X^* \setminus E$ is connected and locally connected. Let $f \in \widetilde{\mathcal{A}}(E)$ be non-critical and let $\varepsilon\colon E \to (0,\infty)$ be a continuous positive valued function. Then there exists a global non-critical holomorphic function $F \in \mathcal{O}(X)$ such that we have $|F(p) - f(p)| < \varepsilon(p)$ for each $p \in E$.*

Since a closed set with empty interior is always semi-admissible, this is just the case where $E = S$, the following corollary is an immediate consequence of the main theorem.

Corollary. *Let $E \subseteq X$ be a closed set with empty interior, such that $X^* \setminus E$ is connected and locally connected. Let $f \in \mathcal{C}(E)$ be a continuous complex valued function on E and let $\varepsilon\colon E \to (0, \infty)$ be a continuous positive valued function. Then there exists a global non-critical holomorphic function $F \in \mathcal{O}(X)$ such that we have $|F(p) - f(p)| < \varepsilon(p)$ for each $p \in E$.*

Note that, as we have seen above, $X^* \setminus E$ being connected and locally connected is a necessary condition to achieve even uniform approximation. Thus, closed sets E with empty interior on which Carleman approximation by non-critical functions is possible are precisely those where $X^* \setminus E$ is connected and locally connected.

Let us now sketch the proof of the main theorem. A precise proof can be found in [6].

Sketch of proof. First we construct a exhaustion $\{K_n\}_{n \in \mathbb{N}_0}$ of X by compact sets without holes such that for each $n \in \mathbb{N}_0$ and $\lambda \in \Lambda$ we have

(i) $h(E \cup K_n) = \emptyset$

(ii) If $H_\lambda \cap K_n \neq \emptyset$, then $H_\lambda \subseteq K_n$.

This can be achieved, since the family of compacts $\{H_\lambda\}_{\lambda \in \Lambda}$ is locally finite and pairwise disjoint. Here we also use the fact that $X^* \setminus E$ is connected and locally connected. For $n \in \mathbb{N}_0$, we define $E_n = E \cup K_n$ and note that these sets are semi-admissible and without holes. For a given sequence of positive values $\{\tilde{\varepsilon}_n\}_{n \in \mathbb{N}_0}$ we will inductively construct a sequence of functions $\{f_n\}_{n \in \mathbb{N}_0}$ such that

(1) $f_n \in \widetilde{\mathcal{A}}(E_n)$ and f_n in non-critical,

(2) $\|f_n - f_{n-1}\|_{E_{n-1} \cap K_{n+1}} < \tilde{\varepsilon}_{n-1}$,

(3) $f_n = f_{n-1}$ on $E \setminus K_{n+1}$.

By choosing the values $\{\tilde{\varepsilon}_n\}_{n \in \mathbb{N}_0}$ sufficiently small, the functions $\{f_n\}_{n \in \mathbb{N}_0}$ will converge uniformly on compacts to a globally defined non-critical holomorphic function F which has the desired approximation property.

The main part of the proof lies in the induction step. So suppose we have already constructed $f_{n-1} \in \widetilde{\mathcal{A}}(E_{n-1})$. We first use Mergelyan's theorem to approximate the function f_{n-1} on $E_{n-1} \cap K_{n+1}$ and obtain a globally defined function h. Since f_{n-1} is non-critical on a neighborhood of the compacts making up E_{n-1}, we can, by approximating well enough, assume that any critical point of h which lies on $E_{n-1} \cap K_{n+1}$ lies on a part of the set with empty interior. By using flow maps of holomorphic vector fields, we can slightly perturb the function h to move all its critical points off of the set $E_{n-1} \cap K_{n+1}$ whilst not changing the function too much. The result is a global function $\tilde{h}$ which is non-critical on an open neighborhood of the set $E_{n-1} \cap K_{n+1}$. We then use a version of Runge's approximation theorem for non-critical holomorphic functions proven by F. Forstnerič in [4] to obtain a global non-critical holomorphic function g which approximates $\tilde{h}$ on $E_{n-1} \cap K_{n+1}$. To summarize, we have the following sequence of approximations

$$f_{n-1} \sim h \sim \tilde{h} \sim g$$

and we end up with a global non-critical holomorphic function g which approximates the function f_{n-1} on $E_{n-1} \cap K_{n+1}$.

Finally, we define a smooth bump function $\chi\colon X \to [0,1]$ which is 1 on a neighborhood of K_n and whose support is contained in the interior of K_{n+1}. We define the function f_n by setting

$$f_n = \chi g + (1-\chi) f_{n-1}.$$

The function f_n is continuous on E_n and holomorphic on a neighborhood of the compact sets making up E_n, since by our choice of χ the function f_n there agrees with either g or f_{n-1}. This proves $f_n \in \widetilde{\mathcal{A}}(E_n)$ and all the other properties follow by construction. □

As another application of the main theorem, let's construct a non-critical entire function with arbitrary asymptotic values.

Recall that $v \in \widehat{\mathbb{C}}$ in an *asymptotic value* of the entire function $F \in \mathcal{O}(\mathbb{C})$, if there exists an unbounded curve $\gamma\colon [0,1) \to \mathbb{C}$ such that $\lim_{t\to 1} F(\gamma(t)) = v$. For a concrete example, consider the complex exponential $z \mapsto e^z$. If one takes γ to be the parametrization of the negative part of the real line, we get $\lim_{t\to 1} e^{\gamma(t)} = 0$, so 0 is an asymptotic value of $z \mapsto e^z$.

Using the main theorem gives us the following proposition.

Proposition 1. *Let $\{v_m\}_{m\in\mathbb{Z}}$ be a sequence of complex numbers. Then there exists a non-critical entire function $F \in \mathcal{O}(\mathbb{C})$ such that v_m is an asymptotic value of F for each $m \in \mathbb{Z}$.*

Proof. The proof is pretty straightforward, we just need to apply the main theorem in the right setting. For $m \in \mathbb{Z}$ let $l_m = \{x + im \mid x \geq 1\}$. Define $E = \bigcup_{m\in\mathbb{Z}} l_m$ and note this set is semi-admissible, has no hole and it enjoys the BEH property. Next, we define the continuous function $f\colon E \to \mathbb{C}$ to be the constant function v_m on the line l_m. Finally, we take $\varepsilon(z) = \frac{1}{\mathrm{Re}(z)}$ and note it is a well defined positive valued continuous function on E. Applying the main theorem yields a non-critical entire function $F \in \mathcal{O}(\mathbb{C})$ such that for any $m \in \mathbb{Z}$ we have $|F(z) - v_m| < \varepsilon(z) = \frac{1}{\mathrm{Re}(z)}$ on l_m. By taking the limit $z \to \infty$ along l_m, this inequality implies that v_m is indeed an asymptotic value of F. □

By choosing $\{v_m\}_{m\in\mathbb{Z}}$ to be a dense set in $\mathbb{C}$, we even have the following corollary.

Corollary. *There exist a non-critical entire function $F \in \mathcal{O}(\mathbb{C})$ such that its asymptotic values are dense in $\mathbb{C}$.*

The existence of non-critical entire functions with arbitrary asymptotic values already follows from the work of L. Boc Thaler in [1]. The benefit of the approach presented here is that we can prescribe the behavior of such a function much more explicitly.

References

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