FOUR COLOR THEOREM AND BEYOND

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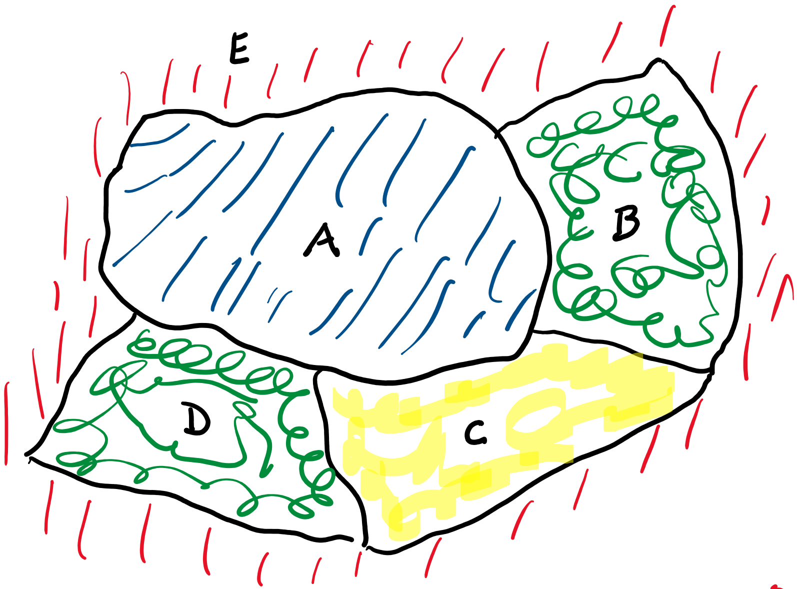
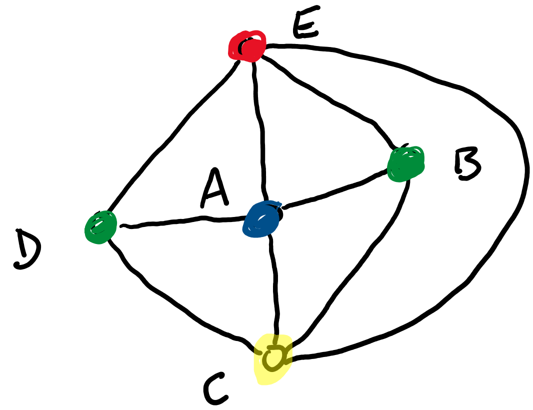
Four Colors Suffice

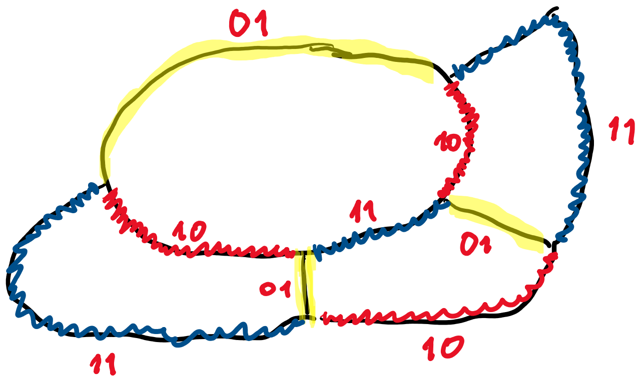




Guthrie 1852

Appel & Haken
1976

Robertson, Seymour
Sanders, Thomas 1996



FOUR-COLOR THEOREM

Every (loopless) planar graph is 4-colorable.

Asked by F. Guthrie in 1850s (map printing)

de Morgan gave it mathematical formulation.

Two false "proofs" in 1890s

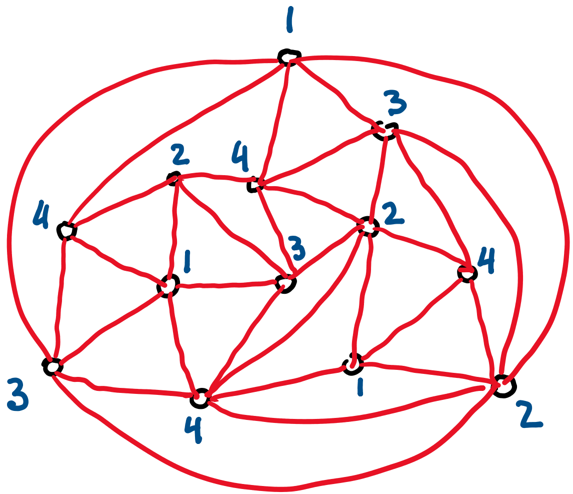
(1) Kempe's "proof" gave useful technique (Kempe change)

(2) Tait's "proof" was based on a wrong assumption that dual graphs of triangulations are always Hamiltonian.

THEOREM (Tait): 4CT is equivalent to the statement that (3-connected) cubic planar graphs are 3-edge-colorable.

WHAT LIES BEYOND THE FOUR COLOR THEOREM?

<u>THEOREM</u>. Every (loopless) planar graph is 4-colorable.



Is this a coincidence?

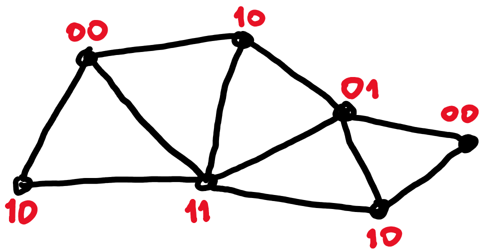
Or are there deeper reasons why the 4CT holds?

Is this really just "the tip of an eisberg"?

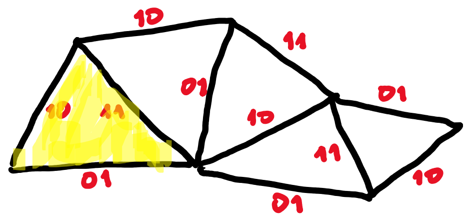
THREE VIEWS ON 4-COLORINGS

Colors can be viewed as the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$

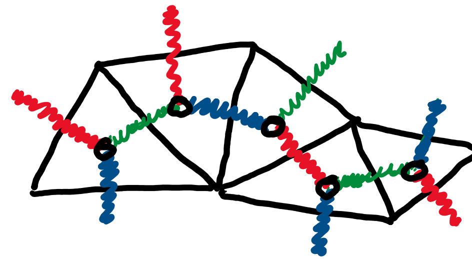
$\{00, 01, 10, 11\}$



4-coloring



Grünbaum coloring



3-edge-coloring (nowhere zero 4-flow)

BEYOND THE FOUR COLOR THEOREM

Graphs on surfaces

Heawood (1890)

Ringel & Youngs (1968)

Albertson (1981)

4CT planar graphs

Grünbaum (1969)

Robertson (1996)

Grötzsch (1960s)

Getting rid of topology

Tutte Flow Conjectures

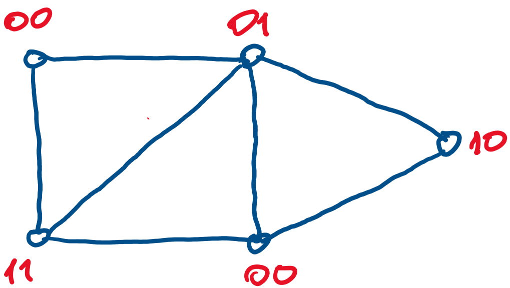
- 3-flow Conjecture
- 4-flow Conjecture
- 5-flow Conjecture

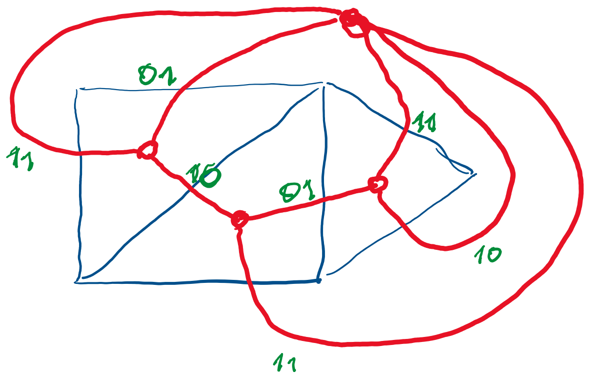
Algorithms

COLORING-FLOW DUALITY (TUTTE, 1960s)

Theorem. A (connected) planar graph is k-colorable $\iff$ its dual graph admits a nowhere-zero k-flow.

Consider 4-coloring of a planar graph with colors in the 4-element group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$





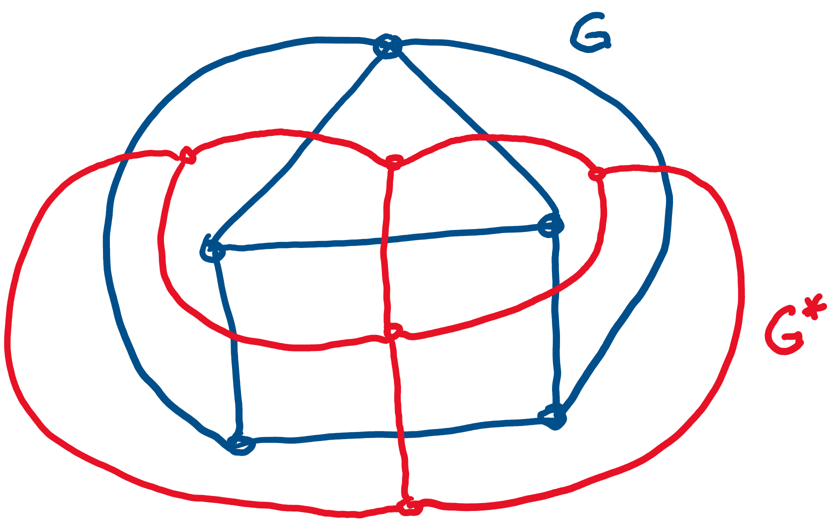
Nowhere-zero 4-flow

FOUR-COLOR THEOREM

Every (loopless) planar graph is 4-colorable.

Grötzsch Theorem (1958). Planar graphs without cycles of length 3 (and 1) are 3-colorable.

Tutte proposed generalization of these results via coloring-flow duality.



Tutte's flow conjectures:

Planar graphs are 5-colorable.

5-flow conjecture (1954)
Every bridgeless graph has a 5-flow.

Planar graphs are 4-colorable.

4-flow conjecture (1966) Every bridgeless graph without P_{10}-minor has a 4-flow.

Planar graphs without cycles of length 3 or 1 are 3-colorable.

3-flow conjecture (1972)
Every graph without 1-cuts and 3-cuts has a 3-flow.

Known results

- 6-flow theorem (Seymour, 1981)
- 4-flow conjecture for cubic graphs (1996/2016(?)) (Robertson, Sanders, Seymour, Thomas)

  double-cross

  reduction

  apex
- Weak 3-flow conjecture (L.M. Lovász, Thomassen, Y. Wu, CQ Zhang)
  6-edge-connected $\Rightarrow \mathbb{Z}_3$-connected.

Generalizations

I Albertson's Conjecture (1981)

$\forall g: \exists k = k(g)$: every (loopless) graph on a surface of (Euler) genus g has a set $U \subseteq V(G)$: $|U| \le k$ and $G - U$ is 4-colorable.

In particular, is $k(1) = 3$ and $k(g) = O(g)$?

II Grötzsch Conjecture (~1960)

A 2-connected subcubic graph is 3-edge-colorable if and only if the number of its vertices ~~of~~ degree 2 is different from 1.

is 0 ~ 4CT

is 2 or 3 ~ Apex case

III Grünbaum's Conjecture (1969)

If T is a triangulation of an orientable surface, then its dual graph is 3-connected cubic graph that has a 3-edge-coloring.

- Counterexamples for genus ≥ 5 were found by K.
- The case of the torus is still open.
- K_6 triangulates the projective plane and its dual is P_{10}, which is not 3-edge-colorable.

Albertson's Conjecture (1981)

<u>Conjecture.</u> $\forall g : \exists k :$

every G embedded in a surface of Euler genus g has a set U of $\leq k$ vertices s.t.

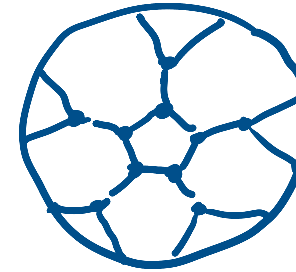
$$\chi(G-U) \leq 4.$$

In particular, $k=3$ for the torus.

K_7 embeds in the torus

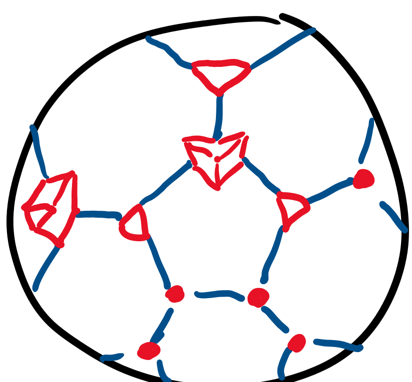
PROJECTIVE PLANE

$K_6 \longleftrightarrow$ Petersen P_{10}

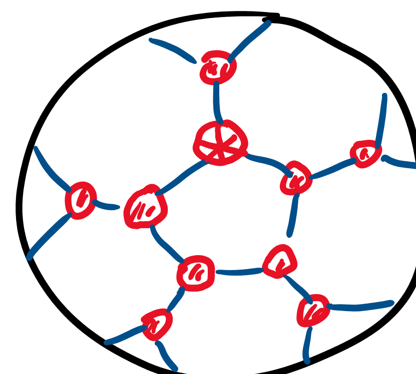


More generally:

G is a **Petersen-like graph** if it can be obtained from the Petersen graph by <u>replacing</u> each vertex by a conn. planar graph (but keep the 15 edges of P_{10}).



Petersen-like cubic graphs are not 3-edge-colorable



Petersen-like (general) graphs do not have NZ 4-flows.

TUTTE 4-FLOW CONJECTURE

A 2-connected cubic graph without P_{10}-minor is 3-edge-colorable.

Strengthening of the T4FC

<u>Theorem.</u> A 2-connected cubic graph embedded in the projective plane is 3-edge-colorable if and only if it is not Petersen-like.

(very long proof, computer-assisted)

<u>Corollary.</u> For a 2-edge-connected graph embedded in the projective plane TFAE:

(i) G has a NZ 4-flow.

(ii) G^* is 5-colorable.

(iii) G is not Petersen-like.

WHAT ABOUT GRAPHS ON THE TORUS ?

① **Coloring**

- $\chi(G) \leq 7$
- $\chi(G) \leq 5$ if ew sufficiently large (4?)
- Albertson's conjecture (with $k \sim 10^{300}$)

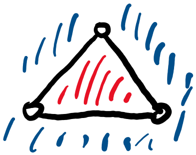
② **Edge-coloring**

- Grünbaum Conjecture
- Strong version of Grünbaum Conjecture (Petersen-like are only examples)
- Infinitely many snarks (dot products of P_{10})

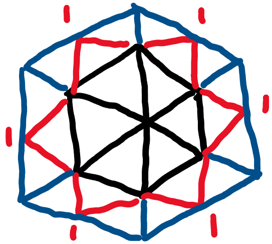
ABOUT THE PROOFS

Planar graphs

- WMA G is a triangulation
- No small separators (3,4,5)



- Reducible configurations
- Make sure that reduction does not give a loop

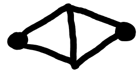


C-reducible

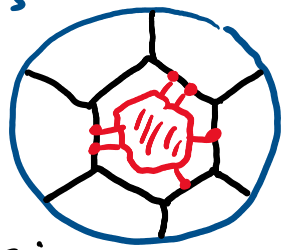
- Small number of discharging rules (103/33/20)

Projective plane

- We work with 3-edge-colorings of cubic graphs.
- Reducible configurations are more complicated, only some planar ones are also good for proj. plane.



- Make sure not to obtain loops or Petersen-like graph
- Emergence of projective configurations



- More than 5000 configurations
- Many discharging rules.

BEYOND THE FOUR COLOR THEOREM

$\boxed{\text{Graphs on surfaces}}$

Heawood (1890)
Ringel & Youngs (1968)

Flat case

$\boxed{\text{4CT planar graphs}}$ ← TBD

Albertson (1981) ← TBD

Grünbaum (1969) ← SODA 2026

TBD

Grötzsch (1960s)

Robertson (1996) ← FOCS 2024

$\boxed{\text{Getting rid of topology}}$

Tutte Flow Conjectures

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- 4-flow Conjecture ← TBD
- 5-flow Conjecture

¿STOC 2026?

$\boxed{\text{Algorithms}}$