How to measure linking?

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Links (in Euclidean space)

Definition (Link)

An oriented (tame) n-**link** L in Euclidean space $\mathbb{R}^3$ is a (tame/PL) embedding $L : \bigsqcup_{i=1}^{n} \mathbb{S}_i^1 \hookrightarrow \mathbb{R}^3$. A mild abuse of notation: $\mathrm{Im}(L) = L$.

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Example

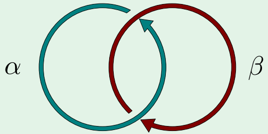

Figure 1: The Hopf link.

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Core idea (due to John Milnor)

Observe ”linking modulo knotting”.

Link homotopy

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Two n-links L and L' are **link homotopic** if there exists an ordered collection of homotopies $\{H^i : \mathbb{S}^1_i \times I \to \mathbb{R}^3\}_{i=1}^n$ between $\{L_i\}$ and $\{L'_i\}$ such that $H^1_t(\mathbb{S}^1_1), \ldots, H^n_t(\mathbb{S}^1_n)$ are pairwise disjoint $\forall t \in I$.

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Link homotopy diagramatically:

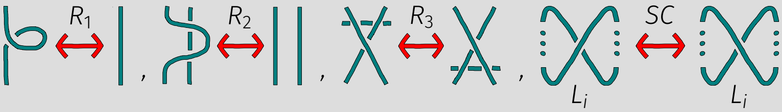

Figure 2: *Link homotopy moves.*

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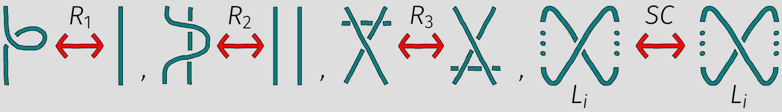

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Important observation!

All knots are link homotopic (to the unknot).

Linking number

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The **linking number** $\mathrm{lk}(L_i, L_j)$ between two components L_i, L_j of link L is the homology class of L_i in $H_1(\mathbb{R}^3 - L_j; \mathbb{Z}) \cong \mathbb{Z}$.

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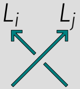
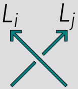
$$\mathrm{lk}(L_1, L_2) = \frac{1}{2}\Big(\big|\{\ \text{[crossing 1]}\ \}\big| - \big|\{\ \text{[crossing 2]}\ \}\big| \Big)$$





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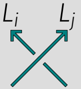
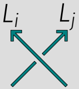
$\mathrm{lk}(L_1, L_2) = \frac{1}{2}\Big(\Big|\Big\{$ 

$\Big\}\Big| - \Big|\Big\{$ 

$\Big\}\Big|\Big)$

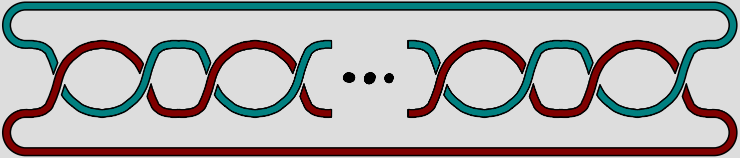
Proposition

The linking number is an invariant wrt. link homotopy.

Classification of two component links

Theorem by Milnor, 1954

The linking number classifies 2-links up to link homotopy. In particular, every 2-link is link homotopic (up to a choice of orientation and number of crossings) to:



Where the linking number fails

Example

The linking number doesn't classify all linking:

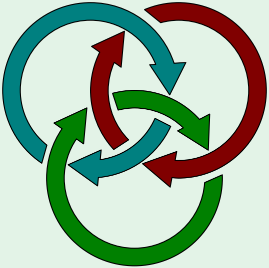


Figure 3: Borromean rings.

Beyond the linking number: Milnor's numerical invariants

Definition (Milnor's $\bar{\mu}$ invariants)

Milnor's $\bar{\mu}$ invariants are the first non-vanishing coefficients of the Magnus expansions of words representing preferred parallels in quotients of the knot group by elements of its lower central series. They are defined (wrt. link homotopy) for finite tuples of pairwise disjoint integers representing the different components of a link: $\bar{\mu}(I) = \bar{\mu}(i_1 \dots i_p)$.

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Milnor's $\bar{\mu}$ invariants detect higher order linking. They take the form of integers.

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Example

The linking number is a length two $\bar{\mu}$ invariant.

Classification of three and four component links

Theorem by Milnor, 1954

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Open problem

Classify k-links up to link homotopy for $k \geq 5$.

Classification of almost trivial links

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Almost trivial n-links are classified up to link homotopy by the $(n-2)!$ invariants

$$\left\{\bar{\mu}\left(1^{\omega}\ldots(n-2)^{\omega}(n-1)n\right) \;\middle|\; \omega \in S_{n-2} \text{ arbitrary permutation}\right\}.$$

Almost trivial links with three components

Example

Almost trivial 3-links are classified by only one number. In particular, each is link homotopic (up to a choice of orientation and number of crossings) to:

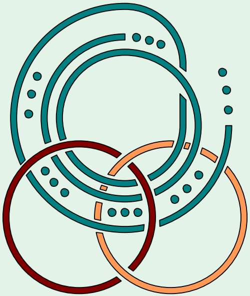

Figure 4: Universal form of almost trivial 3-links.

Examples of almost trivial links

Example

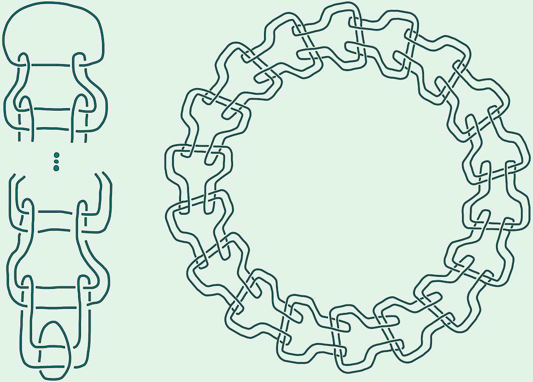

Figure 5: Two almost trivial links

Future work

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- define new invariants for link homotopy,
- give a link homotopy classification of ≤ 4 component links in (more) general 3-manifolds.