Diameter estimates of Kähler manifolds

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This is a joint work with Vincent Guedj and Henri Guenancia.

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where $\|\xi\|_{g(x)} := \sqrt{g(x)(\xi,\xi)}, \quad x = \gamma(t) \in M$ and $\xi = \dot{\gamma}(t) \in T_x M$.

The distance between two points $x, y \in M$, is defined by

$$d_g(x,y) := \inf\{\ell_g(\gamma); \gamma \in C^1([0,1], M), \gamma(0) = x, \gamma(1) = y\},$$

where $C^1([0,1], M)$ be the set of smooth curves of M.

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In particular M is compact by Hopf-Rinow theorem.
A uniform upper bound on the diameter of a sequence of Riemannian manifolds together with a uniform lower bound on the Ricci curvature allows the application of the Gromov compactness theorem.

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Observation : Hermitian metrics always exist on a complex manifold.

Definition

A Hermitian metric h on X is a Kähler metric if its fundamental form $\omega = \omega_h$ is d-closed i.e. $d\omega = 0$. We call ω a Kähler form (or metric) on X and (X, ω) a Kähler manifold.

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In a local chart (U, z) of X, we have

$$h_{|U} = \sum_{1 \leq j,k \leq n} h_{j\bar{k}} dz_j \otimes d\bar{z}_k \longleftrightarrow \omega_{|U} = \frac{i}{2} \sum_{1 \leq j,k \leq n} h_{j\bar{k}} dz_j \wedge d\bar{z}_k.$$

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This expression is independent on the local complex coordinates and defines a (global) real smooth d-closed $(1,1)$-form $\mathrm{Ric}(\omega)$ on X.

Moreover if $\tilde{\omega}$ is another Kähler form on X then

$$\mathrm{Ric}(\omega) - \mathrm{Ric}(\tilde{\omega}) = dd^c \log(\tilde{\omega}^n/\omega^n),$$

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A fundamental problem in Kähler geometry is the converse problem.

Calabi conjecture (1954-1957): Let η be a real smooth d-closed $(1,1)$-form $\eta \in c_1(X)$. Then there exists a Kähler metric ω on X such that $\mathrm{Ric}(\omega) = \eta$ (i.e. one can prescribe the Ricci curvature).

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Let $\eta \in c_1(X)$. By the dd^c-lemma, there $\exists\,! h \in C^\infty(X)$ such that $\mathrm{Ric}(\omega_0) = \eta + dd^c h$ on X and $\int_X e^h \omega_0^n = \int_X \omega_0^n$.

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By the dd^c-lemma there exists $\varphi \in C^\infty(X)$ such that $\omega = \omega_0 + \varphi$.
Then $\mathrm{Ric}(\omega) = \eta$ iff $\varphi \in C^\infty(X)$ with $\omega_0 + dd^c\varphi > 0$ is a solution to the complex Monge-Ampère equation

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S.T. Yau (1978) proved the following fundamental theorem, solving the Calabi conjecture.

Theorem [Yau78]

Let ω_0 *be a fixed Kähler metric on* X *and* $f > 0$ *be a positive smooth density on* X *such that* $\int_X f\omega_0^n = \int_X \omega_0^n$.

Theorem [Yau78]

Let ω_0 be a fixed Kähler metric on X and $f > 0$ be a positive smooth density on X such that $\int_X f\omega_0^n = \int_X \omega_0^n$.
Then $\exists$! $\varphi \in C^\infty(X)$ such that $\omega := \omega_0 + dd^c\varphi > 0$ and $(\omega_0 + dd^c\varphi)^n = f\omega_0^n$ with $\sup_X \varphi = 0$.

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An important consequence of the Calabi conjecture is the following.

Corollary

Let X be a compact Kähler manifold such that $c_1(X) = 0$ (Calabi-Yau manifold). Then any Kähler class Ω on X contains a unique Ricci-flat Kähler metric i.e. $\omega \in \Omega$ and $Ric(\omega) = 0$.

Kolodziej's uniform estimates

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S. Kolodziej (1998) gave a new proof of Yau's C^0 a priori estimate using Pluripotential Theory. Moreover he proved the following theorem.

Theorem [Kolodziej08]

Let (X, ω_X) be a compact Kähler manifold of dimension n. Let $0 \leq f \in L^1(X)$ be a density such that $\int_X f\omega_X^n = \int_X \omega_X^n$. Assume that $f \in L^p(X)$ with $p > 1$ and $\|f\|_{L^p(X)} \leq A < +\infty$. Then there exists a unique weak continuous weak solution $\varphi \in PSH(X, \omega_X) \cap C^0(X)$ to the (degenerate) complex Monge-Ampère equation

$$(\omega_X + dd^c\varphi)^n = f\omega_X^n, \ \max_X \varphi = 0.$$

Moreover $\|\varphi\|_{L^\infty(X)} \leq C(p, n, A, \omega_X)$.

Here the Monge-Ampère measure $(\omega_X + dd^c\varphi)^n$ is defined in the sense of Bedford-Taylor (1976).

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Let $f_\omega := \omega^n / \omega_X^n$. Let $\varphi = \varphi_\omega \in C^\infty(X)$ be the Monge-Ampère potential of the volume form ω^n i.e. $\omega = \omega_X + dd^c \varphi$ and

$$(\omega_X + dd^c \varphi)^n = f \omega_X^n, \quad \max_X \varphi = 0.$$

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Let $f_\omega := \omega^n / \omega_X^n$. Let $\varphi = \varphi_\omega \in C^\infty(X)$ be the Monge-Ampère potential of the volume form ω^n i.e. $\omega = \omega_X + dd^c \varphi$ and

$$(\omega_X + dd^c \varphi)^n = f \omega_X^n, \ \ \max_X \varphi = 0.$$

Let m_φ be the modulus of continuity of φ in the metric space (X, d_{ω_X}).
Then we have

- If $\|f_\omega\|_{L^p(X)} \leq A$ with $p > 1$, then $\forall\, \alpha \in]0, 2/(nq+1)[$,

$$m_\varphi(r) \leq C\, r^\alpha,$$

where $c = C(\alpha, p, A, \omega_X) > 0$ ([Kolodziej [Kol08],[DDGKPZ14]).

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- If $N_p(f_\omega) := \int_X f_\omega |\log f_\omega)|^p \omega_X^n \leq A$ with $p > n$ then

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- If $I_p(f_\omega) := \int_X f_\omega |\log f_\omega|^n [\log(\log(f+3)]^p \omega_X^n \leq A$, with $p > n$, then

$$(K) \qquad m_\varphi(r) \leq C[\log(-\log r)]^{1-p/n},$$

  where $C = C(p, A, \omega_X) > 0$ (Guedj-Guenancia-Z. [GGZ23]).

Main results

Ou first main result is as follows.

Theorem A ([GGZ23])

Let (X, ω_X) *be a compact Kähler manifold of dimension* n*. Let* $\mathcal{K} \subset \mathcal{K}(X)$ *be a compact subset of the Kähler cone. Then for any Kähler metric* ω *such that* $[\omega] \in \mathcal{K}$ *and the density* $f_\omega := \omega^n / \omega_X^n$ *of its volume form satisfies the integrability condition* (K) *with* $p > 2n$ *i.e.*

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we have $diam(X, \omega) \leq C_1 = C_1(A, p, n, \mathcal{K}, \omega_X)$. More precisely, for any $0 < \gamma < p/2 - n$, there exists a constant $C_2 = C_2(A, p, \gamma, \mathcal{K}, \omega_X) > 0$ such that for any $(x, y) \in X^2$ with $d_{\omega_X}(x, y) \leq 1$,

$$d_\omega(x, y) \leq C_2 \left[\log(2 + |\log d_{\omega_X}(x, y))|\right]^{-\gamma}.$$

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Uniform estimates on the diameter of (X, ω) were obtained by :

- (A) Fu-Guo-Song ('20) assuming that :
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- (B) and (C) allow degeneration of the Kähler class $[\omega]$.
- (C) gives remarkable estimates on the Riemannian Green function and a non-collapsing estimates of the volumes of balls of (X, ω), without using any lower bound on the Ricci curvature.

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Remarks :

- Condition (i) (resp. (i')) implies (1) by Kolodziej's a priori estimates.
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Remarks :

- Condition (i) (resp. (i')) implies (1) by Kolodziej's a priori estimates.
- Condition (2) does not imply (1).
- There are examples where (2) is satisfied and the diameters are uniformly bounded while the potentials are not.

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Proposition C ([GGZ23])

Assume that $\int_0^1 \frac{\sqrt{m_\varphi(t)}}{t} dt < +\infty$. *Then* $\exists\ C = C(X, \omega_X) > 0$ *such that*

$$d_\omega(x, y) \leq C\, m_1(d_{\omega_X}(x, y)),\ \forall (x, y) \in X^2,$$

where $m_1(r) := \int_0^r \frac{\sqrt{m_\varphi(t)}}{t} dt$.

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Remarks.

- If $f_\omega \in L^p$ with $p > 1$, the Dini type condition is satisfied.
- If $N_p(f_\omega) < +\infty$, the Dini type condition is satisfied only if $p > 3n$.
- If $I_p(f_\omega) < \infty$, the Dini type condition is never satisfied.

So Theorem A does not follow from Proposition C.

Sketch of the proof of Theorem A :

We assume for simplicity that $X = \mathbb{P}^n$, $\omega_X = \omega_{FS}$ is the Fubini-Study metric and $\omega = \omega_{FS} + dd^c \varphi > 0$, where $\varphi = \varphi_\omega$ is the Monge-Ampère potential i.e. $(\omega_{FS} + dd^c \varphi)^n = f_\omega \omega_X^n$.

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Goal : Fix $x_0 \in X$ and $x \in X$ and estimate the distance function $\rho(x) := d_\omega(x, x_0)$, using the condition (2).

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Let $y \longmapsto g_x(y)$ be the pluricomplex Green function with logarithmic pole at x on the complex projective space $\mathbb{P}^n$.

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Recall that $\rho := d_\omega(\cdot, x_0)$ and $x \in X$ is fixed. Then

$$\rho(x) = \int_X \rho(y)(\omega_{FS} + dd^c_y g_x)^n, \; 0 = \rho(x_0) = \int_X \rho(y)(\omega_{FS} + dd^c_y g_{x_0})^n.$$

We then write

$$\rho(x) = \int_X \rho(y)[(\omega_{FS} + dd^c_y g_x)^n - \omega^n_{FS}] - \int_X \rho(y)[(\omega_{FS} + dd^c_y g_{x_0})^n - \omega^n_{FS}].$$

It's enough to estimate the following integral (x is fixed):

$$I := \int_X \rho(y)[(\omega_{FS} + dd^c_y g_x)^n - \omega^n_{FS}].$$

Set $\omega_{g_x} := \omega_{FS} + dd^c g_x$ and observe that

$$(\omega_{FS} + dd^c g_x)^n - \omega^n_{FS} = dd^c g_x \wedge \sum_{k=0}^{n-1} \omega^k_{g_x} \wedge \omega_{FS}^{n-k-1}.$$

Hence $I = \sum_{k=0}^{n-1} I_k$, where for $0 \leq k \leq n-1$

$$\begin{aligned} I_k &:= \int_X \rho dd^c g_x \wedge \omega_{g_x}^k \wedge \omega_{FS}^{n-k-1} \\ &= \int_X dg_x \wedge d^c \rho \wedge \omega_{g_x}^k \wedge \omega_{FS}^{n-k-1} \text{ (Stokes formula).} \end{aligned}$$

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The most singular term is when $k = n-1$ i.e.

$$I_{n-1} = \int_X dg_x \wedge d^c \rho \wedge \omega_{g_x}^{n-1}.$$

Step 2: Introduce a positive weight and use Cauchy-Schwarz inequality

Then

$$|I_{n-1}(x)|^2 \leq \left(\int_X \chi''(g_x) dg_x \wedge d^c g_x \wedge \omega_{g_x}^{n-1}\right) \tag{3}$$

$$\times \left(\int_X \chi''(g_x)^{-1} d\rho \wedge d^c \rho \wedge \omega_{g_x}^{n-1}\right), \tag{4}$$

where $\chi : \mathbb{R}^- \longrightarrow \mathbb{R}$ is a smooth convex increasing function such that $\chi'(-\infty) = 0$ and $\chi'(0) \leq 1$ to be chosen later.

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Observe then that $\chi(g_x)$ is ω_{FS}-psh in X and

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Then the first term in the RHS of the inequality (3) can be estimated by

$$\int_X [\omega_{FS} + dd^c \chi(g_x)] \wedge \omega_{g_x}^k \wedge \omega_{FS}^{n-k-1} = \int_X \omega_{FS}^n = 1.$$

It remains to estimate the second term defined by

$$J_{n-1}(x) := \int_X \psi_x d\rho \wedge d^c\rho \wedge \omega_{g_x}^{n-1}, \ \ \psi_x := \chi''(g_x)^{-1}.$$

Step 3 : Use Yang Li observation

Since $\rho = d_\omega(\cdot, x_0)$ is Lipschitz w.r.t. d_ω, then $\|\nabla \rho\|_\omega \leq 1$ a.e. and then $d\rho \wedge d^c\rho \leq \omega$ (by Y.Li [Li21]). Recall that $\omega := \omega_{FS} + dd^c\varphi$.

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Again, it's enough to treat the second term which can be written as

$$J''_{n-1}(x) := \int_X \psi_x \, dd^c\varphi \wedge \omega_{g_x}^{n-1} = \int_X \varphi \, dd^c\psi_x \wedge \omega_{g_x}^{n-1}.$$

Now recall that $\psi_x := \chi''(g_x)^{-1}$ and set $h(t) := \chi''(t)^{-1}$ so that $\psi_x = h(g_x)$ and

$$dd^c\psi_x = h'(g_x)dd^c g_x + h''(g_x)dg_x \wedge d^c g_x. \quad (5)$$

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$$J''_{n-1}(x) = \int_X \varphi h'(g_x)dd^c g_x \wedge \omega_{g_x}^{n-1} + \int_X \varphi h''(g_x)dg_x \wedge d^c g_x \wedge \omega_{g_x}^{n-1}$$

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Step 4 : Choose of the weight function and use the estimate on m_φ

Set $\chi(t) := t[\log(B - t)]^{-\gamma}$ for $t < 0$, where $\gamma > 0$ is small enough and $B > 1$ large enough so that χ is increasing convex on $\mathbb{R}^-$ and $\chi'(0) \le 1$.

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$$h(-s) \sim s(\log s)^{1+\gamma}, h'(-s) \sim (\log s)^{1+\gamma}, h''(-s) \sim \frac{(\log s)^{\gamma}}{s}.$$

We can assume from the beginning that $\varphi(x) = 0$. Then we have for any $y \in X$ close to x, $-g_x(y) \sim \log d_{FS}(x, y)$ and

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It remains to estimate the third term

$$K_3 := \int_X \varphi h''(g_x) dg_x \wedge d^c g_x \wedge \omega_{g_x}^{n-1}.$$

Step 5 : Use estimates on the Green function

A simple computation shows that

$$\omega_{g_x} \leq e^{-2g_x}\omega_{FS}, \text{ and } dg_x \wedge d^c g_x \leq e^{-2g_x}\omega_{FS}.$$

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We have established a uniform upper bound of the diameter of (X, ω).
More refined estimates allow to prove the precise estimate of the distance d_ω in terms of the distance d_{ω_X}.

Thank you for your attention