Global regularity in the $\bar{\partial}$-Neumann problem — a new approach and generalizations.

Dmitri Zaitsev *zaitsev@maths.tcd.ie*

Trinity College Dublin

Global regularity and the $\bar\partial$-Neumann problem

Global regularity

A fundamental question for (boundary value problem for) systems of PDEs is finding conditions for *global regularity*, i.e. when (weak) solutions are smooth whenever the data are smooth.

$\bar\partial$-Neumann problem

$$(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)u = f, \quad u \in \operatorname{dom} \bar\partial\bar\partial^* \cap \operatorname{dom} \bar\partial^*\bar\partial,$$

where $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ is the *complex Laplacian*, u, f are $(0, q)$-forms, $\bar\partial\colon \Omega^{0,q} \to \Omega^{0,q+1}$ the $\bar\partial$-operator and $\bar\partial^*\colon \Omega^{0,q+1} \to \Omega^{0,q}$ the adjoint in L^2.

Formulated in the 50s by Spencer as a means to generalize Hodge theory.

Important application: regularity of proper holomorphic maps

Global regularity in $\bar\partial$-Neumann problem $\Longrightarrow$ regularity of Bergman projection (Condition R by Bell-Ligocka) $\Longrightarrow$ smooth boundary extension of proper holomorphic maps, generalizing Fefferman's theorem.

Finite type and Property (P_q)

In his influential work, Catlin developed elaborate geometric and potential-theoretic techniques to tackle the problem of global regularity of the $\bar{\partial}$-Neumann problem for pseudoconvex bounded domains in $\mathbb{C}^n$ of finite D'Angelo type.

In particular, Catlin's potential-theoretic *property* (P) inspired vast applications and continuing research on *compactness* in the $\bar{\partial}$-Neumann problem, that is known to imply global regularity by Kohn-Nirenberg.

Definition (Catlin for $q = 1$ in a global, Sibony in a local context, extended by Fu-Straube for $q > 1$.)

A compact subset $K \subset \mathbb{C}^n$ is said to satisfy *property* (P_q) (called simply property (P) for $q = 1$), where $1 \le q \le n$ is an integer, if for every $C > 0$, there exists an open neighborhood U of K in $\mathbb{C}^n$ and a real C^2 function $\lambda \colon U \to \mathbb{R}$ such that $0 \le \lambda \le 1$ and the *sum of any q eigenvalues* of the complex hessian matrix $(\lambda_{z_j \bar{z}_k})_{j,k=1}^n$ is greater than C.

Property (P_q) has found vast applications, including:

1. *compactness* of the $\bar\partial$-Neumann operator (i.e. the inverse of the complex Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$) (Catlin, Straube, Christ-Fu, Harrigton);
2. regularity of the Szegö projection (Boas);
3. estimates for the Bergman kernels (Bell-Boas), the Szegö kernels and for the Bergman, Kobayashi metrics (Chen-Fu);
4. compactness of the complex Green operator (i.e. the inverse of the *Kohn Laplacian* $\Box_b = \bar\partial_b\bar\partial_b^* + \bar\partial_b^*\bar\partial_b$) (Nicoara, Raich, Straube, Khanh-Pinton-Zampieri, Biard);
5. the Diederich-Fornaess exponent and non-existence of Stein domains with Levi-flat boundaries (Sibony, Shaw-Fu);
6. approximation by plurisubharmonic and by holomorphic functions in neighborhoods of domain's closure, constructions of bounded exhaustion functions, existence of Stein neighborhood bases (Sibony);
7. Holomorphic *Morse inequalities* and *eigenvalue asymptotics* for the complex Laplacian (Fu-Jacobowitz).

From finiteness conditions to global regularity

finiteness conditions $\Longrightarrow$ generalized stratification

generalized stratification $\Downarrow$ property (P_q)

property (P_q) $\Downarrow$ compactness

global regularity $\Longleftarrow$ compactness

global regularity $\Uparrow$ condition R

Examples of finiteness conditions and relations between them:

finite D'Angelo type $\Longrightarrow$ finite regular type

finite regular type $\Downarrow$ finite tower multitype

We shall *focus on the first implication shown in red*, while other implications in black are well-understood.

Catlin's argument for property (P) is not well-understood

The only available source leading to property (P) for finite type

There seems to be a lack in understanding for Catlin's argument which still, after 40 years, remains the only available source to obtain property (P) (and hence compactness, global regularity of the $\bar{\partial}$-Neumann operator and condition R) for general smoothly bounded pseudoconvex domains of finite type in any dimension.

Despite its significance and numerous requests by experts in the field there had not been any alternative exposition or simplification of the techniques covering the general case.

One of the present work's goals is to fill this gap and provide a new approach using a different multitype invariant, called the tower multitype, and a different set of new techniques.

New finiteness conditions

Another goal is this work is to provide a range of new finiteness conditions implying global regularity. Our inspiration here comes from the recent breakthrough work by Huang-Yin on the Bloom conjecture comparing *3 different approaches to (regular) types* going back to Kohn — respectively based on:

1. *contact orders* with complex submanifolds;
2. *iterated commutators* in subbundles to span the tangent space;
3. orders of nonzero *derivatives of the Levi form* along subbundles.

The exact relationship between contact orders and types for pseudoconvex hypersurfaces remains a difficult problem and part of the Bloom's and D'Angelo's conjectures, both still open in general, with only recent substantial progress made by Huang-Yin that initiated further research by W. Chen, Y. Chen, X. Huang, W. Yin, P. Yuan.

Without pseudoconvexity, the above approaches are not equivalent, as shown by examples of Bloom, D'Angelo, Huang-Yin.

Notation: local defining functions

1. *smooth* means always C^∞;
2. $S \subset \mathbb{C}^{n+1}$ is a smooth real hypersurface, $n \geq 1$;
3. a *local defining function* r of S in a neighborhood U of p is any smooth real function with

$$S \cap U = \{r = 0\}$$

and $dr \neq 0$ at every point of U;

Notation: tangent bundles

1. TS is the *real tangent bundle*;
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the *complexified tangent bundle*;
3. $H^{10}S = \{X \in \mathbb{C}TS : \partial r(X) = 0\}$ is the $(1,0)$ bundle;
4. $H^{01}S = \{X \in \mathbb{C}TS : \bar{\partial} r(X) = 0\}$ is the $(0,1)$ bundle;
5. $HS = \mathrm{Re} H^{10}S = \mathrm{Re} H^{01}S \subset TS$ is the *complex tangent bundle*;
6. We have the standard relations:

$$H^{01}S = \overline{H^{10}S}, \quad \mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S}.$$

Theorem (special case, more general refined version to follow)

Let $D \subset \mathbb{C}^n$ be a pseudoconvex bounded domain, *$S = \partial D$, $1 \le q \le n-1$ an integer. Assume $\forall p \in S$, at least one of the following conditions holds:*

1. *there does not exist a* formal holomorphic immersion
$$\gamma \colon (\mathbb{C}^q, 0) \to (\mathbb{C}^n, p)$$
that is tangent to S of infinite order, i.e. $r \circ \gamma \equiv 0$ formally;
2. *there does not exist a* smooth complex subbundle *$E \subset H^{10}S$ of rank q in a neighborhood of p such that*
$$[L^m, \dots, [L^2, L^1] \dots](p) \in \mathbb{C}HS$$
holds for all $m \ge 2$ and all vector fields $L^m, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E})$;
3. *there does not exist a* smooth complex subbundle *$E \subset H^{10}S$ of rank q in a neighborhood of p such that*
$$L^m \cdots L^3 \partial r([L^2, L^1])(p) = 0$$
holds for all $m \ge 2$ and all vector fields $L^m, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E})$.

Then S satisfies property (P_q), *hence global regularity holds.*

Finiteness conditions: origin and other work

Finiteness conditions as in the above theorem and notions of type in the study of the $\bar{\partial}$-Neumann problem go back to Kohn, Bloom-Graham, Bloom and D'Angelo.

In 1974, Kohn formulated a conjecture that a condition similar to (2) is *necessary and sufficient for subellipticity*. Remarkably, the above theorem implies that Kohn's condition in the conjecture implies *compactness and global regularity rather than subellipticity*.

Commutator conditions in the $\bar{\partial}$-Neumann problem have been studied by Kohn, Derridj, Tartakoff, Fefferman, Harrington-Raich and others.

Implications for finite regular and singular contact types

Recall: regular and singular contact types

1. the regular q-type of S at $p \in S$ is the supremum of contact orders at p of S with germs of *holomorphic immersions* $\gamma\colon (\mathbb{C}^q, 0) \to (\mathbb{C}^n, p)$;
2. singular q-type is the supremum of *contact orders* at p of S with germs of *complex q-dimensional subvarieties*.

Condition (1) in the above theorem is satisfied when S is of finite regular (contact) q-type at p, in particular, also when S is of *finite (singular) q-type* (or simply finite D'Angelo type for $q = 1$). Indeed, if (1) were violated for a *formal holomorphic immersion* $\gamma\colon (\mathbb{C}^q, 0) \to (\mathbb{C}^n, p)$ tangent to S of infinite order, truncating γ to arbitrary high orders would yield complex q-dimensional submanifolds of arbitrarily high contact orders with S, *contradicting the finiteness of the regular q-type*.

In comparison with Catlin's approach, our tower multitype is based on *minimization of numbers of vector fields* defining certain distinguished nested sequences of $(1,0)$ subbundles, called towers, rather than maximization of rational weights over choices of local holomorphic coordinates. We don't use weighted truncations of geometric objects and their coordinate representations. Instead, further new techniques include:

1. real and complex formal (Nagano type) orbits O, V for certain special subbundles defined in terms of derivatives of the Levi form;
2. formal CR property for orbits O that we call formal Huang-Yin property in analogy with its convergent variant crucial in their work;
3. relative contact orders of real hypersurfaces with pairs (O, V) $\mathrm{ord}(r) = \sup\{k : r \in I(O)^k + I(V)\}$, where r is a defining function;
4. supertangent vector fields having higher than expected relative contact orders, satisfying a certain Lie algebra property;
5. infiniteness of relative contact orders with (O, V) when O is complex-tangential — a *formal variant* of a key step by Diederich-Fornæss in their proof of Kohn's ideal termination in the real-analytic case.

Formal orbits and contact orders

To relate contact orders with subbundles, we introduce:

Definition

Let $S \subset \mathbb{C}^n$ be a smooth real submanifold and $p \in S$ a fixed point.

1. The *complex formal orbit* $\mathcal{O}_E^{\mathbb{C}}(p)$ of a smooth (real or complex) subbundle $E \subset \mathbb{C}TS$ at p is the *complex formal variety* given by the complex *formal power series ideal*

$$I = \{f \in \mathbb{C}[[z-p]] : L^t \cdots L^1 f(p) = 0,\ L^t, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}),\ t \geq 0\}.$$

2. The *contact order* at p between S and a *formal variety* V given by an ideal $I(V) \subset \mathbb{C}[[z-p]]$ in the ring of formal power series is

$$\max\{k : I_p(S) \subset I(V) + \mathfrak{m}^k\} \in \mathbb{N} \cup \{\infty\},$$

where $I_p(S) \subset \mathbb{C}[[z-p]]$ is the *ideal of formal power series vanishing on S* (in the formal sense) and $\mathfrak{m} \subset \mathbb{C}[[z-p]]$ is the *maximal ideal*.

Definition (subbundle types)

Let $S \subset \mathbb{C}^n$ be a real smooth hypersurface and $E \subset H^{10}S$ a *smooth complex subbundle*.

1. The contact type $a(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ of E at p is the contact order at p between S and the complex formal orbit $\mathcal{O}_E^{\mathbb{C}}(p)$;
2. The (commutator) type $t(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ of E at p is
$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), [L^m, \ldots, [L^2, L^1] \ldots](p) \notin \mathbb{C}HS\}.$$
3. The Levi (form) type $c(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ of E at p is
$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), L^m \cdots L^3 \partial r([L^2, L^1])(p) \neq 0\}.$$

With this definition, conditions (2) and (3) in theorem can be restated as:

(2) there *does not exist a smooth complex subbundle* $E \subset H^{10}S$ of rank q in a neighborhood of p with $t(E,p) = \infty$, i.e. of *infinite commutator type*;

(3) there *does not exist a smooth complex subbundle* $E \subset H^{10}S$ of rank q in a neighborhood of p with $c(E,p) = \infty$, i.e. of *infinite Levi type*.

Definition (Dual forms and special subbundles)

Let $S \subset \mathbb{C}^n$ be a smooth real hypersurface with a complex contact form θ.

1. The *θ-dual form* of an (ordered) list of complex vector fields
$$L^t, \ldots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S}), \quad t \geq 1,$$
is the complex 1-form $\omega_{L^t,\ldots,L^1;\theta}$ on $H^{10}S$ defined for $L \in \Gamma(H^{10}S)$ by
$$\begin{cases} \omega_{L^1;\theta}(L_p) := \theta([L, L^1])(p) & t \\ \omega_{L^t,\ldots,L^1;\theta}(L_p) := (L f_{L^t,\ldots,L^1;\theta})(p), \;\; f_{L^t,\ldots,L^1;\theta} := \mathrm{Re}(L^t \cdots L^3 \theta([L^2, L^1])) & t \end{cases}$$

2. A complex subbundle $E \subset H^{10}S$ is *special* if it can be defined by
$$E = \{\xi \in H^{10}S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0\}, \;\; \omega_1 \wedge \cdots \wedge \omega_l \neq 0 \text{ on } \Lambda^l H^{10}S,$$
where each ω_j, $j = 1, \ldots, l$, is the θ-dual 1-form $\omega_j = \omega_{L_j^{t_j},\ldots,L_j^1}$ for some $t_j \geq 1$ and $L_j^{t_j}, \ldots, L_j^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$.

Refined finiteness conditions

Theorem (refined version: each of 3 conditions implies (P_q))

Property (P_q) still holds under finiteness conditions stated only for special subbundles $E \subset H^{10}S$ *of rank* $\geq q$*, i.e. when for every* $p \in S$*, at least one of the following conditions holds:*

(1′) *there does not exist a special smooth complex subbundle* $E \subset H^{10}S$ *of rank* $\geq q$ *in a neighborhood of* p *with* $a(E, p) = \infty$*, i.e. of infinite contact type, and whose complex formal orbit* $\mathcal{O}^{\mathbb{C}}_E(p)$ *is regular, i.e. defined by a* formal manifold ideal*;*

(2′) *there does not exist a special smooth complex subbundle* $E \subset H^{10}S$ *of rank* $\geq q$ *in a neighborhood of* p *with* $t(E, p) = \infty$*, i.e. of infinite commutator type;*

(3′) *there does not exist a special smooth complex subbundle* $E \subset H^{10}S$ *of rank* $\geq q$ *in a neighborhood of* p *with* $c(E, p) = \infty$*, i.e. of infinite Levi type.*

Definition (Towers and tower multitype)

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface with complex contact form θ.

1. A complex 1-form ω defined on $H^{10}S$ is called *E-dual of order $t \in \mathbb{N}_{\geq 2}$*, where $E \subset H^{10}S$ is a complex subbundle, if it is θ-dual of a list of $(t-1)$ complex vector fields
$$L^{t-1}, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E})).$$
2. A *tower on S of multi-order* $(t_1, \dots, t_n) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$ is a nested sequence of complex subbundles
$$H^{10} = E_0 \supset \dots \supset E_m, \quad 0 \leq m \leq n,$$
such that $t_{m+1} = \dots = t_n = \infty$, and for each $k = 1, \dots, m$, one has $t_k \in \mathbb{N}_{\geq 2}$ and there exists an E_{k-1}-dual form ω_k of order t_k with
$$E_k = E_{k-1} \cap \{\omega_k = 0\}, \quad \omega_k|_{E_{k-1}} \neq 0,$$
3. The *tower multitype* of S at $p \in S$ is the CR invariant
$$\mathcal{T}(p) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$$
defined as the *lexicographically minimum multi-order* $(t_1, \dots, t_n)$ of a tower on a neighborhood of p in S.

Finite tower multitype implies (P_q) for all q

The following theorem provides new finiteness conditions in terms of the tower multitype guaranteeing property (P_q):

Theorem (main theorem for q-finite tower multitype)

Let $D \subset \mathbb{C}^n$ be a pseudoconvex bounded domain with smooth boundary $S = \partial D$, and $1 \leq q \leq n-1$. Assume that for every boundary point $p \in S$, the tower multitype $\mathcal{T}(p) = (t_1, \ldots, t_n)$ is q-finite, i.e. satisfies

$$\#\{k : t_k = \infty\} < q,$$

where $\#$ is the number of elements. Then S satisfied property (P_q).

In particular, we obtain the special case:

Corollary

Let $D \subset \mathbb{C}^n$ be a pseudoconvex bounded domain. Assume that for every boundary point $p \in S = \partial D$, all entries of the tower multitype $\mathcal{T}(p) = (t_1, \ldots, t_n)$ are finite. Then S satisfied property (P_q) for all q.

Finiteness conditions — summary

The scheme of implications between various conditions can be illustrated by the diagram:

$$\begin{array}{ccccccc}
(1) & \Longrightarrow & (1') & \searrow & & & \\
(2) & \Longrightarrow & (2') & \Longrightarrow & q\text{-finite tower multitype} & \Longrightarrow & (P_q) \\
(3) & \Longrightarrow & (3') & \nearrow & & &
\end{array}$$

Here $(1')$, $(2')$, $(3')$ are conditions of respectively finite contact, commutator and Levi type for *special subbundles*.

Example (Finite tower multitype but all (1′),(2′),(3′) are violated)

Let $S \subset \mathbb{C}^3_{z_1,z_2,w}$ be the pseudoconvex real hypersurface defined by $r = 0$,

$$r = -2\text{Re}w + |z_1 z_2|^2 + f(z_1, z_2),$$

where f is any smooth real function in $\mathbb{C}^2$ that is *strictly plurisubharmonic away from* 0 and vanishes to infinite order at 0. Then, for $p \neq 0$, S is strictly pseudoconvex, hence the tower multitype $\mathcal{T}(p) = (2, 2)$.

We claim that $\mathcal{T}(0) = (4, 4)$. Indeed, fixing contact form $\theta = \partial r$ and vector field

$$L = \partial_{z_1} + \partial_{z_2} + a(z_1, z_2)\partial_w \in H^{10}S$$

we have $\omega_1|_{E_0} \neq 0$ for $E_0 = H^{10}S$ and the 1-form $\omega_1 = \omega_{\overline{L},L,\overline{L};\theta}$, i.e.

$$\omega_1 = d\text{Re}\overline{L}\partial r([L, \overline{L}) = d\text{Re}\overline{L}L\overline{L}r = c(dz_1 + dz_2), \quad c \neq 0.$$

Hence $t_1 = 4$ is the minimum choice for the first entry of the multiorder (the number of vector fields) tower multitype $\mathcal{T} = (t_1, t_2) = (4, t_2)$.

Example showing that finite tower multitype is more general than conditions $(1')$,$(2')$,$(3')$ — continuation

Example (continuation)

Setting

$$E_1 = E_0 \cap \{\omega_1 = 0\}, \quad E_2 = 0,$$

we obtain a tower $E_0 \supset E_1 \supset E_2$ of the minimum multi-order $(t_1, t_2) = (4, 4)$. Thus the *tower multitype has all entries finite* at all points.

On the other hand, since S is tangent of infinite order to the complex lines $w = z_1 = 0$ and $w = z_2 = 0$, condition (1) in the main theorem is violated for $q = 1$. Taking the subbundle $E \subset H^{10}S$ defined by $dz_2 = 0$ and $L' = \partial_{z_1} + a'(z_1, z_2)\partial w$, it can be seen that conditions (2) and (3) are also violated. Moreover, E can be seen to be special, hence also conditions $(1')$-$(3')$ in the refined version of the main theorem are violated.

Recall: from finiteness conditions to global regularity

finiteness conditions $\Longrightarrow$ generalized stratification

generalized stratification $\Downarrow$ property (P_q)

property (P_q) $\Downarrow$ compactness

global regularity $\Longleftarrow$ compactness

global regularity $\Uparrow$ condition R

Examples of finiteness conditions and relations between them:

finite D'Angelo type $\Longrightarrow$ finite regular type

finite regular type $\Downarrow$ finite tower multitype

We shall focus on the first implication shown in red, while other implications in black are well-understood.

Generalized stratifications with convexity properties

First implication – existence of generalized stratifications with certain *convexity properties*:

Definition (generalizing regular domains by Catlin-Diederich-Fornaess)

A hypersurface $S \subset \mathbb{C}^{n+1}$ is countably q-regular ($1 \le q \le n$) if it is a *countable disjoint union* $S = \cup_{k=1}^{\infty} S_k$ of locally closed subsets $S_k \subset S$ ("strata") such that for each k and $p \in S_k$, there exists a CR submanifold $M \subset S$ satisfying the following properties:

1. M contains an open neighborhood of p in S_k (in relative topology);
2. $\dim_{\mathbb{C}}(H^{10}_x M \cap K^{10}_x) < q$ for all $x \in M$, where $K^{10}_x \subset H^{10}_x S$ is the *kernel of the Levi form* of S.

When $q = 1$ and S is pseudoconvex, (2) means the Levi form of S is *positive definite along $H^{10}M$*. This allows constructions of *bounded* local *weight functions aka barriers* with *large complex hessians* on strata S_k as $C(r + \sum_j r_j^2)$, where r (resp. r_j) are local defining functions of S (resp. M). (Countable 1-regularity $\iff$ empty Levi core by Dall'Ara-Mongodi.)

Bounded barriers with large complex hessians (BBLH)

The *complex hessian* of a real function λ is the hermitian quadratic form

$$H_\lambda(X) := \sum \lambda_{z_j \bar{z}_k} X_j \overline{X}_k.$$

Property (P_1) for $K \iff$ the existence of BBLH on $K \iff$ the existence of functions λ bounded by 1 in a neighborhood of A with arbitrarily large complex hessian.

Sibony's B-regularity theory

Local existence of BBLH for strata S_k implies global existence of BBLH for their compact countable unions.

Precise formulation (Sibony, Fu-Straube)

Let $S = \cup_k S_k \subset \mathbb{C}^n$ be a compact disjoint union of countably many compact subsets. Assume for each k, each $p \in S_k$ has a compact neighborhood in S_k satisfying (P_q). Then S satisfies (P_q).

Countable 1-regularity for finite tower multitype: sketch

1. θ-dual function of $(L^t, \dots, L^1)$ is $f_{L^t,\dots,L^1;\theta} := \mathrm{Re}(L^t \cdots L^3\theta([L^2, L^1]))$;
2. For a tower $H^{10} = E_0 \supset \dots \supset E_m, \quad E_k = E_{k-1} \cap \{\omega_k = 0\}$, where $\omega_k = \omega_{L_k^{t_k-1},\dots,L_k^1;\theta}$, collect all θ-dual functions $\{f_{L_k^{t_k-1},\dots,L_k^1;\theta} : t_k \geq 2\}; \quad f_k := f_{L_k^{t_k-1},\dots,L_k^1;\theta}, \; k > l$,
3. the joint zero set M of those functions is a smooth *CR submanifold*;
4. the kernel K^{10} of the Levi form tensor λ_θ of S is transversal to M in the sense that it satisfies $H^{10}M \cap K^{10} \subset E_m$.
5. *finite tower multitype* $\implies$ $m = n$, $E_n = 0$ $\implies \mathrm{H}^{10}M \cap K^{10} = \{0\}$;
6. the tower multitype *level set* satisfies $\{p' \in U : \mathcal{T}(p') = \mathcal{T}(p)\} \subset M$.
7. generalized stratification with convexity propreties using level sets of $\mathcal{T}$ as strata:

$$S = \bigcup_{(t_1,\dots,t_n)\in(\mathbb{N}_{\geq 2}\cup\{\infty\})^n} \{p : \mathcal{T}(p) = (t_1, \dots, t_n)\}.$$

Finite Levi type implies finite tower multitype

Definition

The Levi type $c(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$$\min\{t \geq 2 : \exists L^t, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), L^m \cdots L^3 \partial r([L^2, L^1])(p) \neq 0\},$$

where r is a local defining function of S.

Proposition

Assume $c(E,p) < \infty$ for any smooth subbundle $E \subset H^{10}S$ of rank 1. Then S has finite tower multitype at p.

Hence under the assumptions of the proposition, S is *countably 1-regular* $\implies (P_1)$.

Finite commutator type implies finite tower multitype

1. Suffices to show: Levi type is finite for all special subbundles of rank ≥ 1;
2. Assume by contradiction, there is a special subbundle E of rank ≥ 1, which is of *infinite Levi type* at p;
3. real formal orbit is given by the ideal $I(O) = \{f \in \mathbb{R}[[x-p]] : L^t \cdots L^1 f(p) = 0,\ L^t, \dots, L^1 \in E \cup \overline{E},\ t \geq 0\}$;
4. The orbit O is *formally complex-tangential* to S in the sense that $D_O \subset HS \mod I(O)$, where D_O is the Lie algebra of formal vector fields tangent to O;
5. then $T_pO \subset H_pS \implies T_pS$ is not spanned by commutators from E $\implies$ contradiction with the finite commutator type assumption.

Finite regular type implies finite tower multitype: sketch

1. Suffices to show: Levi type is finite for all special subbundles of rank ≥ 1;
2. Assume by contradiction, there is a special subbundle E of rank ≥ 1, which is of *infinite Levi type* at p;
3. $\implies$ the real orbit O of $E \cup \overline{E}$ is formally CR;
4. $\implies$ the complexification V of O is a formal complex submanifold;
5. desired contradiction is obtained from a formal variant of a result of Diederich-Fornaess:

Theorem $(*)$

Let S be pseudoconvex, *$O \subset V \subset \mathbb{C}^n$ as above $\implies$ V is tangent to S of infinite order.*