A new Poincaré type rigidity phenomenon with applications

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Plan of the talk

- Part I: Two notions of flatness
- Part II: Poincaré type rigidity (**main part**)
- Part III: Applications of the rigidity results in studying flat hypersurfaces

Part I: Two notions of flatness

- Bergman logarithmically flatness
- Obstruction flatness

Bergman logarithmically flatness

Let $G \subseteq \mathbb{C}^n$: smoothly bounded strongly pseudoconvex domain.

Write $G = \{z \in \mathbb{C}^n : r(z, \overline{z}) > 0\}$ with a smooth defining function r.

Fefferman showed that the Bergman kernel K_G of G obeys that the following asymptotic expansion on G:

$$K_G = \frac{\phi}{r^{n+1}} + \psi \log r.$$

Here $\phi, \psi \in C^\infty(\overline{G})$, and $\phi|_{\partial G} \neq 0$.

Bergman logarithmically flatness

With Fefferman's expansion of the Bergman kernel K_G of G:

$$K_G = \frac{\phi}{r^{n+1}} + \psi \log r,$$

We can make the following definition:

Let $M \subseteq \partial G$ be an open subset of ∂G. We say M is **Bergman logarithmically flat** if ψ vanishes to the infinite order along M.

Bergman logarithmically flatness

By the localization property of the Bergman kernel (cf. Fefferman, Boutet de Monvel and Sjöstrand, Kaneko, Engliš, Huang-Li):

- Bergman logarithmic flatness only depends on the local CR geometry of the boundary;
- We can define Bergman logarithmic flatness for any strongly pseudoconvex CR hypersurface (in a complex manifold).

Bergman logarithmically flatness

Definition:

A strongly pseudoconvex CR hypersurface Σ (in a complex manifold) is called **Bergman logarithmically flat** if at every $p \in \Sigma$, both (1) and (2) hold:

(1) There exists a small smoothly bounded strongly pseudoconvex domain G which has Σ as part of boundary near p;

(2) The coefficient function ψ in the Fefferman expansion of K_G vanishes to the infinite order along Σ near p.

Recall: $K_G = \frac{\phi}{r^{n+1}} + \psi \log r.$

Remark. By the localization property of the Bergman kernel, the definition does not depend on the choice of G in (1).

Bergman logarithmically flatness

Example:

Consider the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$.

The Bergman kernel of $\mathbb{B}^n$ is given by $K_{\mathbb{B}^n} = \frac{n!}{\pi^n} \frac{1}{(1-|z|^2)^{n+1}}$.

$\Rightarrow$ The sphere $\partial\mathbb{B}^n$ is Bergman logarithmically flat.

Since Bergman logarithmically flatness is a (local) CR invariant, any **spherical** CR hypersurface is Bergman logarithmically flat.

Obstruction flatness

Now it leads to a natural question:

Question: Are there Bergman logarithmically flat CR hypersurfaces that are non-spherical?

Answer:

- In $3-$dimensional case: No. That is,

  Bergman logarithmically flat $\Leftrightarrow$ spherical.
  (by Graham-Burns, Boutet de Monvel)

- In 5 or higher dimensional case: Yes.

  Engliš-Zhang: $\exists$ a compact CR hypersurface (in some complex manifold) of dimension $2n+1$ for $n \geq 2$ that is
  - Bergman logarithmically flat,
  - transversally symmetric,
  - non-spherical.

Next we discuss obstruction flatness

$\Omega \subset \mathbb{C}^n$: smoothly bounded, strongly pseudoconvex domain.

- The existence of a complete Kähler–Einstein metric on Ω is governed by the following Dirichlet problem of the Monge-Ampère (MA) equation:

$$\begin{cases} J(u) := (-1)^n \det \begin{pmatrix} u & u_{\overline{z_k}} \\ u_{z_j} & u_{z_j\overline{z_k}} \end{pmatrix} = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{1}$$

- If u is a solution to (1), then $-\partial\overline{\partial} \log u$ gives a complete Kähler-Einstein metric on Ω.

Obstruction flatness

Solutions to the Dirichlet problem of the MA equation:

$$\begin{cases} J(u) = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega. \end{cases} \tag{2}$$

- **Fefferman defining function.**
  Fefferman: $\exists$ an approximate solution $\rho \in C^\infty(\overline{\Omega})$ to (2) such that $J(\rho) = 1 + O(\rho^{n+1})$.
  Note: Such ρ is unique up modulo $O(\rho^{n+2})$.

- **The Cheng-Yau solution.**
  Cheng-Yau: $\exists$ a unique solution $u \in C^\infty(\Omega)$ to (2).

Obstruction flatness

Lee-Melrose proved:.

The Cheng-Yau solution u has the asymptotic expansion:

$$u \sim \rho \sum_{k=0}^{\infty} \eta_k \left(\rho^{n+1} \log \rho\right)^k, \tag{3}$$

where each $\eta_k \in C^\infty(\overline{\Omega})$ and ρ is a Fefferman defining function.

$\Rightarrow$ in general the Cheng-Yau solution $u \in C^{n+2-\varepsilon}(\overline{\Omega})$ but does not extend C^∞ across the boundary $\partial\Omega$.

Definition: If the Cheng-Yau solution u extends C^∞ across an open piece M of the boundary $\partial\Omega$, then we say M obstruction flat.

By Graham's work:

Obstruction flatness is a local CR invariant. It only depends on the local CR geometry of the boundary.

Using Graham's work, one can define the obstruction flatness for any strongly pseudoconvex hypersurface (in a complex manifold).

Obstruction flatness

Definition:

A strongly pseudoconvex CR hypersurface (in a complex manifold) Σ is called **obstruction flat** if at every $p \in \Sigma$, both (1) and (2) hold:

(1) There exists a small smoothly bounded strongly pseudoconvex domain Ω which has Σ as part of boundary near p;

(2) Cheng–Yau solution u on Ω extends C^∞ across Σ near p.

Remark. By the work of Graham, the definition does not depend on the choice of Ω in (1).

Obstruction flatness

Example:

Consider the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$.

The Cheng-Yau solution on $\mathbb{B}^n$ is given by $1 - |z|^2 \in C^\infty(\overline{\mathbb{B}^n})$.

$\Rightarrow$ The sphere $\partial\mathbb{B}^n$ is obstruction flat.

Obstruction flatness

Example:

Consider the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n : |z| < 1\}$.

The Cheng-Yau solution on $\mathbb{B}^n$ is given by $1 - |z|^2 \in C^\infty(\overline{\mathbb{B}^n})$.

$\Rightarrow$ The sphere $\partial\mathbb{B}^n$ is obstruction flat.

Since obstruction flatness is a (local) CR invariant, any **spherical** CR hypersurface is obstruction flat.

Obstruction flatness

Now it leads to a natural question:

Question: Must obstruction flat CR hypersurfaces be spherical?

Answer:

- Locally: no, by the work of Graham. That is, locally there are non-spherical ones.

- As oppose to the local setting,

  Ebenfelt: a compact obstruction flat 3-dimensional CR hypersurface with transverse symmetry must be spherical.

  Later, Curry-Ebenfelt proved more general results.

Obstruction flatness

Much less is known in higher dimensions. In particular

Question:
Let M be a compact obstruction flat CR hypersurface with transverse symmetry of dimension ≥ 5.

Must M be spherical?

Much less is known in higher dimensions. In particular

Question:
Let M be a compact obstruction flat CR hypersurface with transverse symmetry of dimension ≥ 5.

Must M be spherical?

The answer is NO! More precisely,

Obstruction flatness

In 5 **and higher dimensions,**

More precisely, by Ebenfelt-X.-Xu,

$\exists$ a CR hypersurface (in some complex manifold) of dimension $2n+1$ for each $n \geq 2$ that is

- compact,
- transversally symmetric,
- obstruction flat,
- non-spherical.

Two notions of flatness

Summary:

Easily, we always have:

Sphericity $\Rightarrow$ Bergman log. flatness + Obstruction flatness.

- In the 3−dimensional case:

  Bergman log. flatness $\Leftrightarrow$ Sphericity;

  Obstruction flatness $\not\Rightarrow$ Sphericity.

  Obstruction flatness + Compactness + Transverse symmetry $\Rightarrow$ Sphericity.

Two notions of flatness

Summary:

- In 5 or higher dimensions:

  Bergman log. flatness $\not\Rightarrow$ Sphericity;

  Obstruction flatness $\not\Rightarrow$ Sphericity.

  Moreover, $\exists$ explicit examples showing

  Bergman log. flatness + Compactness + Transverse symmetry $\not\Rightarrow$ Sphericity.

  Obstruction flatness + Compactness + Transverse symmetry $\not\Rightarrow$ Sphericity.

Nevertheless, the following question seemed unclear in 5 and higher dimensions:

Q: Is it possible to construct two hypersurfaces such that they both are

- compact,
- transversally symmetric,
- obstruction flat (respectively, Bergman logarithmically flat),
- non-spherical,

while possessing distinct local CR structures?

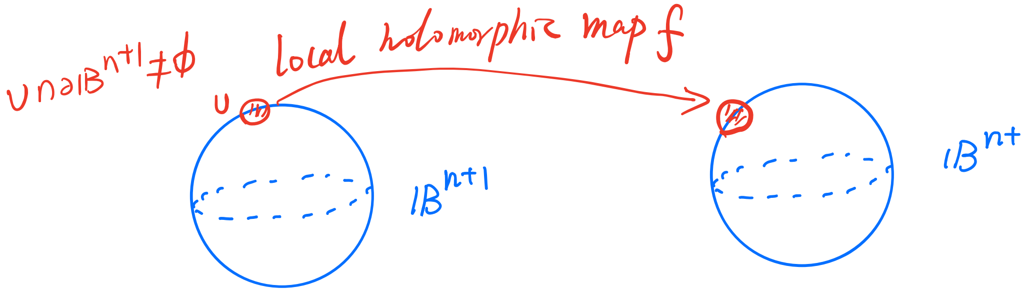
Part II: Poincaré type rigidity

The following celebrated classical theorem was due to Poincaré for $n = 1$, and Tanaka, Chern-Moser, Alexander for $n \geq 2$.

Write $\mathbb{B}^m = \{z \in \mathbb{C}^m : |z| < 1\}$.

Theorem 1. *Let U be a connected open subset of $\mathbb{C}^{n+1}$, $n \geq 1$, with $U \cap \partial\mathbb{B}^{n+1} \neq \emptyset$. Let F be a nonconstant holomorphic map from U to $\mathbb{C}^{n+1}$. If F maps $U \cap \partial\mathbb{B}^{n+1}$ to $\partial\mathbb{B}^{n+1}$, then F must extend to a holomorphic automorphism of $\mathbb{B}^{n+1}$ (in particular, F is linear fractional).*

Poincaré Rigidity:



If f: nonconstant and $f(U \cap \partial \mathbb{B}^{n+1}) \subseteq \partial \mathbb{B}^{n+1}$,

$\Rightarrow$ f extends to a holomorphic automorphism of $\mathbb{B}^{n+1}$.

Poincaré type rigidity:

- (Extension part) “local map between the boundaries” extends to “global map between the domains”,
- (Rigidity part) Besides, often the resulted global map possesses some special properties (e.g., biholomorphism).

Poincaré type rigidity

Numerous Poincaré type rigidity phenomenons were discovered, for the balls and bounded symmetric domains (the source and target domains can even be different dimensional).

See work by Webster, Faran, Forstnerič, Huang, Baouendi, Ebenfelt, Mok-Ng, Zaitsev-Kim, etc.

See also lots of related work by D'Angelo, Lebl, Pinchuk, Nemirovski, Shafikov, Lamel, Mir, etc.

We will present a new Poincaré type rigidity phenomenon which extends Theorem 1 via a different viewpoint.

To illustrate, we first observe $\mathbb{B}^{n+1}$ can be regarded as **a disk bundle over the lower dimensional ball** $\mathbb{B}^n$.

Poincaré type rigidity

More precisely,

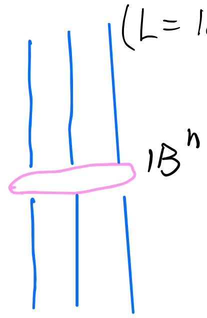
let $L = \mathbb{B}^n \times \mathbb{C}$ be the trivial line bundle over $\mathbb{B}^n$.

Note: Recall a holomorphic line bundle over a contractible Stein manifold is always trivial.

We equip L with the Hermitian metric $h(z, \overline{z}) = (1 - |z|^2)^{-1}$ where $z \in \mathbb{B}^n$. Then

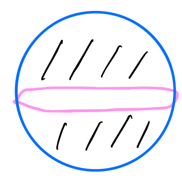
- (L, h) is a negative line bundle;
- The negative of its Chern class $-c_1(L, h)$ induces the standard hyperbolic metric on $\mathbb{B}^n$.
- In this way, the disk bundle $D(L, h) = \mathbb{B}^{n+1}$;
  circle bundle $C(L, h) =$(an open dense subset of) $\partial\mathbb{B}^{n+1}$.

$(L = \mathbb{B}^n \times \mathbb{C},\ h = (1-|z|^2)^{-1})$



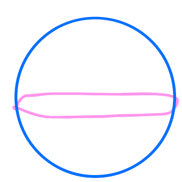
Note: The disc bundle D of (L, h):

$$D(L,h):\quad |\xi|^2 h(z,\bar{z}) < 1, \quad z \in \mathbb{B}^n$$
$$\Leftrightarrow\quad |\xi|^2 (1-|z|^2)^{-1} < 1, \quad z \in \mathbb{B}^n$$
$$\Leftrightarrow\quad |\xi|^2 + |z|^2 < 1, \quad z \in \mathbb{B}^n$$



RMK: $D(L,h) = \mathbb{B}^{n+1}$

$$C(L,h):\quad |\xi|^2 h(z,\bar{z}) = 1, \quad z \in \mathbb{B}^n$$
$$\Leftrightarrow\quad |\xi|^2 (1-|z|^2)^{-1} = 0, \quad z \in \mathbb{B}^n$$
$$\Leftrightarrow\quad |\xi|^2 + |z|^2 = 1, \quad z \in \mathbb{B}^n$$



RMK: $C(L,h) \subsetneq \partial \mathbb{B}^{n+1}$ is an open, dense subset.

Note:

With this viewpoint, Theorem 1 can now be formulated as follows.

A nonconstant local holomorphic map sending an open piece of the circle bundle $C(L, h)$ to $C(L, h)$ must extend to an automorphism of the disk bundle $D(L, h)$.

Poincaré Rigidity:

Here base domain $\Omega = \mathbb{B}^n$

disk bundle $D(L,h) = \mathbb{B}^{n+1}$

circle bundle $C(L,h) \simeq \partial \mathbb{B}^{n+1}$

$\Rightarrow$ F extends to a (biholomorphic) automorphism of $D(L,h)$

Poincaré type rigidity

Question: What if we replace the base $\mathbb{B}^n$ by polydisk, for simplicity, say bidisk Δ^2?

That is, let $L = \Delta^2 \times \mathbb{C}$ be the trivial line bundle over Δ^2 equipped with, e.g., the Hermitian metric

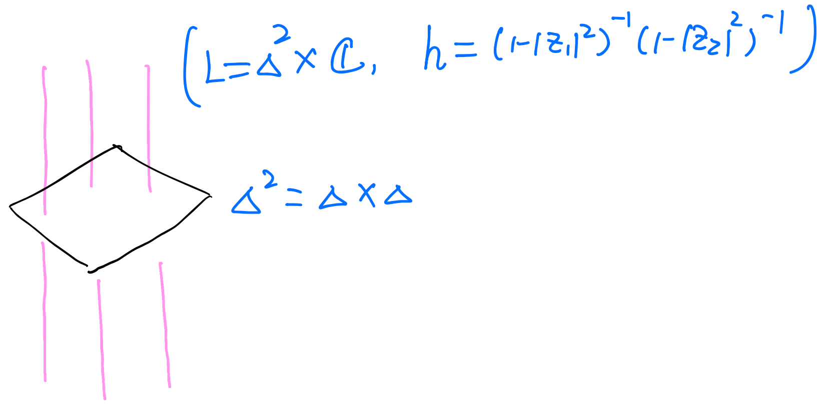
$h(z, \overline{z}) = (1 - |z_1|^2)^{-1}(1 - |z_2|^2)^{-1}$, with $z = (z_1, z_2)$.

Then the corresponding circle and disk bundles are:

$$C(L, h) := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)\}.$$

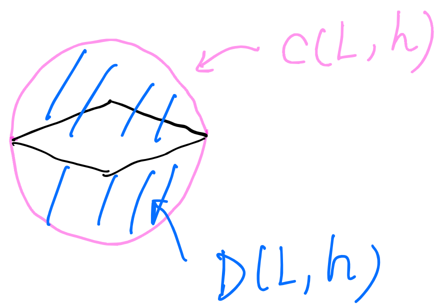
$$D(L, h) := \{(z_1, , z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 < (1 - |z_1|^2)(1 - |z_2|^2)\}.$$

(1)



$(L = \Delta^2 \times \mathbb{C},\quad h = (1-|z_1|^2)^{-1}(1-|z_2|^2)^{-1})$
$\Delta^2 = \Delta \times \Delta$

(2)



$C(L,h)$
$D(L,h)$

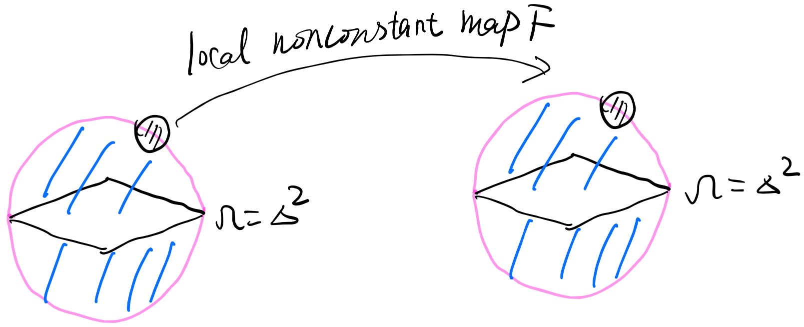
Poincaré type rigidity

With these new circle and disk bundles:

$C(L,h) := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)\}.$

$D(L,h) := \{(z_1, , z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 < (1 - |z_1|^2)(1 - |z_2|^2)\}.$

Question: Does a nonconstant local holomorphic map sending an open piece of $C(L,h)$ to $C(L,h)$ extend to an automorphism of the disk bundle $D(L,h)$?



Q: Does F extend to an automorphism of the disk bundle?

Poincaré type rigidity

With these new circle and disk bundles:

$C(L, h) := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)\}.$

$D(L, h) := \{(z_1, , z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 < (1 - |z_1|^2)(1 - |z_2|^2)\}.$

Question: Does a nonconstant local holomorphic map sending an open piece of $C(L, h)$ to $C(L, h)$ extend to an automorphism of the disk bundle $D(L, h)$?

A subtlety: Comparing to the previous $\Omega = \mathbb{B}^n$ case, $D(L, h)$ does not have smooth boundary. Although $C(L, h)$ is smooth, it is just an open dense subset of $D(L, h)$.

Poincaré type rigidity

With these new circle and disk bundles:

$C(L, h) := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)\}.$

$D(L, h) := \{(z_1, , z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 < (1 - |z_1|^2)(1 - |z_2|^2)\}.$

Question: Does a nonconstant local holomorphic map sending an open piece of $C(L, h)$ to $C(L, h)$ extend to an automorphism of the disk bundle $D(L, h)$?

Spoiler alert: The answer is yes! Moreover, the map is rational (but in general not linear fractional).

Recall:

$$C(L,h) := \{(z_1, z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 = (1 - |z_1|^2)(1 - |z_2|^2)\}.$$

$$D(L,h) := \{(z_1, , z_2, \xi) \in \Delta^2 \times \mathbb{C} : |\xi|^2 < (1 - |z_1|^2)(1 - |z_2|^2)\}.$$

Remark: By work of Webster+Bryant, $C(L,h)$ is **NOT** spherical!

Indeed by combining the work of Webster+Bryant, we have

Theorem: Let $(L, h) \to (M, g)$ be a negative line bundle over a complex manifold of complex dimension at least two inducing a Kähler metric g. Assume that either M is compact or (M, g) has constant scalar curvature. Then $C(L, h)$ is spherical if and only if (M, g) is locally holomorphically isometric to one of

(1) $(\mathbb{B}^n, \lambda\,\omega_{-1})$ for some $\lambda \in \mathbb{R}^+$,

(2) $(\mathbb{CP}^n, \lambda\,\omega_1)$ for some $\lambda \in \mathbb{R}^+$,

(3) $(\mathbb{C}^n, \omega_0)$,

(4) $(\mathbb{B}^l \times \mathbb{CP}^{n-l}, \lambda\omega_{-1} \times \lambda\omega_1)$ for some $\lambda \in \mathbb{R}^+$.

Here ω_c denotes the Kähler metric with constant holomorphic sectional curvature c.

For our case,

$$\left(L = \Delta^2 \times \mathbb{C},\ h = (1-|z_1|^2)^{-1}(1-|z_2|^2)^{-1} \right)$$

$$\downarrow$$

$$\Delta^2$$

Note: the induced metric is the product of two poincaré metrics, which is none of (1) – (4)

$\Longrightarrow$ $C(L,h)$ is NOT spherical.

Webster + Bryant

We next recall: Bounded Symmetric Domains

Poincaré Disk: Hyperbolic space form

The unit disk Δ in $\mathbb{C}$ with Poincaré metric:

$$\omega = \frac{dz \otimes d\bar{z}}{(1-|z|^2)^2}$$

Möbius transformations:

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \qquad a \in \Delta.$$

Properties of Möbius transformations:

- φ_a is a holomorphic isometry: $\varphi_a^*(\omega) = \omega$.
- φ_a is an involution: $\varphi_a \circ \varphi_a = id$.
- φ_a has only one fixed point: $\frac{a}{1+\sqrt{1-|a|^2}}$.

Bounded Symmetric Domains

Definition

A bounded symmetric domain is a bounded domain Ω equipped with a complete Hermitian metric g such that for any point $p \in \Omega$, there exists $\sigma_p : \Omega \to \Omega$ satisfying

- σ_p is a holomorphic isometry: $\sigma_p^*(g) = g$.
- σ_p is an involution: $\sigma_p \circ \sigma_p = id$.
- p is an isolated fixed point of σ_p.

Remark 1.
A bounded symmetric domain Ω is irreducible $\Leftrightarrow \Omega \neq \Omega_1 \times \Omega_2$.

An irreducible bounded symmetric domain is Kähler-Einstein with $\mathrm{Ric} < 0$.

Why bounded symmetric domains?

Remark 2. Bounded symmetric domains are of fundamental important in complex geometry.

Example.

Borel 1963: Every bounded symmetric domain covers a compact complex manifold.

Hano 1957: A bounded homogeneous domain Ω covers a compact complex manifold $\Rightarrow$ Ω is a bounded symmetric domain.

Frankel 1989: A bounded convex domain Ω covers a compact complex manifold $\Rightarrow$ Ω is a bounded symmetric domain.

Irreducible bounded symmetric domains can be classified into:

(1) Cartan's classical domains:

- Type I domain: $D^I_{p,q} := \{z \in M(p,q,\mathbb{C}) : I_p - z\bar{z}^t > 0\}$

  Special case: The unit ball $\mathbb{B}^n$ is of type I $(p = 1, q = n)$.
- Type II domain: $D^{II}_n := \{z \in D^I_{n,n} : z = -z^t\}$
- Type III domain: $D^{III}_n := \{z \in D^I_{n,n} : z = z^t\}$
- Type IV domain: denoted by D^{IV}_n, also called the Lie ball,

  $\{z = (z_1, \cdots, z_n) \in \mathbb{C}^n | z\bar{z}^t < 2 \text{ and } 1 - z\bar{z}^t + \frac{1}{4}|zz^t|^2 > 0\}$.

(2) Two exceptional domains

They are equipped with standard Bergman metrics

$$\omega := \sqrt{-1}\partial\bar{\partial}\log K(z,\bar{z}):$$

Example. Cartan's classical domains:

- Type I: $K(z,\bar{z}) = \mathrm{Vol}^{-1}\cdot\left(\det(I_p - z\bar{z}^t)\right)^{-(p+q)}$.
- Type II: $K(z,\bar{z}) = \mathrm{Vol}^{-1}\cdot\left(\det(I_n - z\bar{z}^t)\right)^{-(n-1)}$.
- Type III: $K(z,\bar{z}) = \mathrm{Vol}^{-1}\cdot\left(\det(I_n - z\bar{z}^t)\right)^{-(n+1)}$.
- Type IV: $K(z,\bar{z}) = \mathrm{Vol}^{-1}\cdot\left(1 - z\bar{z}^t + \frac{1}{4}|zz^t|^2\right)^{-n}$.

Here Vol denotes the volume of the domain in the Euclidean measure.

Generic norms of irreducible bounded symmetric domains:

Let

- Ω: an irreducible bounded symmetric domain
- $K(z, \bar{z})$: Bergman kernel of Ω

Then

$\exists$ a positive integer γ associated to Ω, called the genus of Ω, such that $(\mathrm{Vol} \cdot K(z, \overline{z}))^{-\frac{1}{\gamma}}$ is an irreducible real polynomial.

This polynomial is the generic norm of Ω, denoted by $N(z, \bar{z})$.

Example. Cartan's classical domains:

- Type I: $K(z,\bar{z}) = \mathrm{Vol}^{-1} \cdot \left(\det(I_p - Z\overline{Z}^t)\right)^{-(p+q)}$.

  $\gamma = p+q; \quad N(z,\bar{z}) = (\mathrm{Vol} \cdot K(z,\bar{z}))^{-\frac{1}{\gamma}} = \det(I_p - Z\overline{Z}^t);$

  Special case:

  If $\Omega = \mathbb{B}^n (p=1, q=n)$, then $N(z,\bar{z}) = 1 - |z|^2$.

- Type II: $K(z,\bar{z}) = \mathrm{Vol}^{-1} \cdot \left(\det(I_n - z\bar{z}^t)\right)^{-(n-1)}$.

  $N(z,\bar{z}) = \sqrt{\det(I_n - z\bar{z}^t)};$

- Type III: $K(z,\bar{z}) = \mathrm{Vol}^{-1} \cdot \left(\det(I_n - z\bar{z}^t)\right)^{-(n+1)}$.

  $N(z,\bar{z}) = \det(I_n - z\bar{z}^t);$

- Type IV: $K(z,\bar{z}) = \mathrm{Vol}^{-1} \cdot \left(1 - z\bar{z}^t + \frac{1}{4}|zz^t|^2\right)^{-n}$.

  $N(z,\bar{z}) = 1 - z\bar{z}^t + \frac{1}{4}|zz^t|^2$.

Poincaré type rigidity

Now consider:

- $\Omega = \Omega_1 \times \cdots \times \Omega_s$: a bounded symmetric domain in $\mathbb{C}^n$, where each Ω_i is an irreducible bounded symmetric domain in $\mathbb{C}^{n_i}_{z_i}$ with $n = \sum_{i=1}^{l} n_i$.
- Write N_i for the generic norm of Ω_i.
- $L = \Omega \times \mathbb{C}$ be the trivial line bundle equipped with the Hermitian metric $h(z, \bar{z}) = \prod_{i=1}^{s} \left(N_i(z_i, \overline{z_i})\right)^{-k_i}$ with all $k_i \in \mathbb{N}$.

Then

(1) $-c_1(L, h) = \partial\bar{\partial} \log h$ induces a complete homogeneous Kähler metric ω_Ω on Ω.

(2) Moreover, the corresponding circle and disk bundles are

(2) Moreover, the corresponding circle and disk bundles are

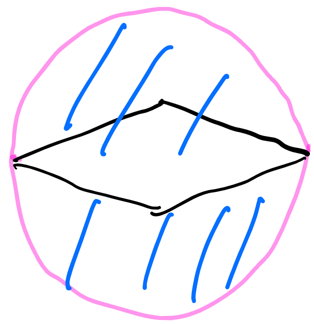
$$C(L,h) := \{(z,\xi) \in \Omega \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^{s} \big(N_i(z_i,\overline{z_i})\big)^{k_i}\} \subseteq \mathbb{C}^{n+1}.$$

$$D(L,h) := \{(z,\xi) \in \Omega \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^{s} \big(N_i(z_i,\overline{z_i})\big)^{k_i}\} \subseteq \mathbb{C}^{n+1}.$$

Special case: If we let $\Omega = \mathbb{B}^n$ (thus $s = 1$) and let $k_1 = 1$, then $D(L,h) = \mathbb{B}^{n+1}$.

Remark. $C(L,h)$ is not compact. Moreover, its (compact) closure in $\mathbb{C}^{n+1}$ is not smooth in general.

Base manifold Ω = a bdd symmetric domain



$\partial\Omega$: NOT smooth in general

(2) Moreover, the corresponding circle and disk bundles are

$$C(L,h) := \{(z,\xi) \in \Omega \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^{s} \left(N_i(z_i,\overline{z_i})\right)^{k_i}\} \subseteq \mathbb{C}^{n+1}.$$

$$D(L,h) := \{(z,\xi) \in \Omega \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^{s} \left(N_i(z_i,\overline{z_i})\right)^{k_i}\} \subseteq \mathbb{C}^{n+1}.$$

Special case: If we let $\Omega = \mathbb{B}^n$ (thus $s = 1$) and let $k_1 = 1$, then $D(L,h) = \mathbb{B}^{n+1}$.

Remark. By work of Webster+Bryant, $C(L,h)$ is never spherical unless $\Omega = \mathbb{B}^n$.

Important CR geometric property of the circle bundle $C(L, h)$:

- Since (L, h) is a negative line bundle, $C(L, h)$ is strongly pseudoconvex.
- $C(L, h)$ is homogeneous.
- $C(L, h)$ is Bergman logarithmically flat and obstruction flat.

  Why:

  (1) By Ahn-Park, the Bergman kernel of $D(L, h)$ is rational. Then by an argument of Ebenfelt-X.-Xu, $C(L, h)$ is Bergman logarithmically flat.

  (2) Since $-c_1(L, h)$ induces a homogeneous metric on Ω, by a theorem of Ebenfelt-X.-Xu, $C(L, h)$ is obstruction flat.

To introduce our main theorem,

we consider two sets of such circle and disk bundles: (I) and (II)

(I)

- $\Omega = \Omega_1 \times \cdots \times \Omega_s$: a bounded symmetric domain in $\mathbb{C}^n$.
- N_i : the generic norm of Ω_i.
- $L = \Omega \times \mathbb{C}$: the trivial line bundle equipped with the metric $h(z, \bar{z}) = \prod_{i=1}^{s} \left(N_i(z_i, \overline{z_i})\right)^{-k_i}$ with all $k_i \in \mathbb{N}$.

Then

(1) $-c_1(L, h) = \partial\overline{\partial} \log h$ induces a complete homogeneous Kähler metric ω_Ω on Ω.

(2) The corresponding circle and disk bundles:

$C(L, h) := \{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^{s} \left(N_i(z_i, \overline{z_i})\right)^{k_i}\} \subseteq \mathbb{C}^{n+1}$.

$D(L, h) := \{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^{s} \left(N_i(z_i, \overline{z_i})\right)^{k_i}\} \subseteq \mathbb{C}^{n+1}$.

(II)

- $\widetilde{\Omega} = \widetilde{\Omega}_1 \times \cdots \times \widetilde{\Omega}_t$: a bounded symmetric domain in $\mathbb{C}^n$.
- $\widetilde{N}_i$ for the generic norm of $\widetilde{\Omega}_i$.
- $\widetilde{L} = \widetilde{\Omega} \times \mathbb{C}$: the trivial line bundle equipped with the metric $\widetilde{h}(z, \bar{z}) = \prod_{i=1}^{t} \left(\widetilde{N}_i(z_i, \overline{z_i})\right)^{-\widetilde{k}_i}$ with all $\widetilde{k}_i \in \mathbb{N}$.

Then

(1) $-c_1(\widetilde{L}, \widetilde{h}) = \partial\bar{\partial} \log \widetilde{h}$ induces a complete homogeneous Kähler metric $\omega_{\widetilde{\Omega}}$ on $\widetilde{\Omega}$.

(2) the corresponding circle and disk bundles:

$$C(\widetilde{L}, \widetilde{h}) := \{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^{t} \left(\widetilde{N}_i(z_i, \overline{z_i})\right)^{\widetilde{k}_i}\} \subseteq \mathbb{C}^{n+1}.$$

$$D(\widetilde{L}, \widetilde{h}) := \{(z, \xi) \in \Omega \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^{t} \left(\widetilde{N}_i(z_i, \overline{z_i})\right)^{\widetilde{k}_i}\} \subseteq \mathbb{C}^{n+1}.$$

Poincaré type rigidity

Theorem

(X., 2023) Let F be a nonconstant local holomorphic map sending an open piece of $C(L, h) \subseteq \mathbb{C}^{n+1}$ to $C(\widetilde{L}, \widetilde{h}) \subseteq \mathbb{C}^{n+1}$. Assume at least one k_i and at least one $\widetilde{k}_i$ equal to 1. Then

(1) The map F extends to a rational biholomorphism from $D(L, h)$ to $D(\widetilde{L}, \widetilde{h})$.

(2) The Kähler manifolds (Ω, ω_Ω) and $(\widetilde{\Omega}, \omega_{\widetilde{\Omega}})$ are holomorphically isometric.

Remark on the proof: Since the (compact) closure of $C(L, h)$ in $\mathbb{C}^{n+1}$ is not smooth in general, typical analytic continuation tools for local holomorphic maps (cf., Pinchuk, Huang-Ji, Shafikov, etc) do not apply.

Poincaré type rigidity

Theorem

(X., 2023) Let F be a nonconstant local holomorphic map sending an open piece of $C(L,h) \subseteq \mathbb{C}^{n+1}$ to $C(\widetilde{L},\widetilde{h}) \subseteq \mathbb{C}^{n+1}$. Assume at least one k_i and at least one $\widetilde{k}_i$ equal to 1. Then

(1) The map F extends to a rational biholomorphism from $D(L,h)$ to $D(\widetilde{L},\widetilde{h})$.

(2) The Kähler manifolds (Ω, ω_Ω) and $(\widetilde{\Omega}, \omega_{\widetilde{\Omega}})$ are holomorphically isometric.

Note. The conclusion fails if the assumption "at least one k_i and at least one $\widetilde{k}_i$ equal to 1" is dropped, as shown by examples.

Poincaré type rigidity

Recall: For the ball $\mathbb{B}^n$, generic norm $N(z,\overline{z}) = 1 - |z|^2$.

Example : Write

$$C_1 := \{(z,\xi) \in \mathbb{B}^n \times \mathbb{C} : |\xi|^2 = (1 - |z|^2)^2\} \subseteq \mathbb{C}^{n+1}.$$

$$C_2 := \{(z,\xi) \in \mathbb{B}^n \times \mathbb{C} : |\xi|^2 = 1 - |z|^2\} \subseteq \mathbb{C}^{n+1}.$$

Then

$F_1(z,\xi) = (z, \sqrt{\xi})$ maps (locally) C_1 to C_2;

$F_2(z,\xi) = (z, \xi^2)$ maps C_2 to C_1.

Next, What does the theorem says the special case when $\Omega = \widetilde{\Omega} = \Delta^n$?

Corollary

Let $(k_1, \cdots, k_n), (\widetilde{k}_1, \cdots, \widetilde{k}_n) \in \mathbb{N}^n$*. Write* $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$*. Set*

$C = \{(z, \xi) \in \Delta^n \times \mathbb{C} : |\xi|^2 = \prod_{i=1}^n (1 - |z_i|^2)^{k_i}\};$ *likewise for* $\widetilde{C}$;

$D = \{(z, \xi) \in \Delta^n \times \mathbb{C} : |\xi|^2 < \prod_{i=1}^n (1 - |z_i|^2)^{k_i}\};$ *likewise for* $\widetilde{D}$.

Assume at least one k_i *and at least one* $\widetilde{k}_i$ *equal to* 1*. Let* F *be a nonconstant local holomorphic map sending an open piece of* C *to* $\widetilde{C}$*. Then*

(1) after a permutation, $(k_1, \cdots, k_n) = (\widetilde{k}_1, \cdots, \widetilde{k}_n)$*.*

(2) F *extends to a rational biholomorphism from* D *to* $\widetilde{D}$*.*

Part III: Applications of the rigidity results in studying flat hypersurfaces

Corollary

(X., 2023) For every $n \geq 2$, $\exists$ a countably infinite family $\mathcal{F}$ of compact real analytic CR hypersurfaces of dimension $2n+1$ such that

(1) Every $M \in \mathcal{F}$ is obstruction flat and Bergman log. flat;

(2) Every $M \in \mathcal{F}$ is locally homogeneous and transversally symmetric;

(3) Local CR structures of hypersurfaces in $\mathcal{F}$ are mutually inequivalent.

Remark. (3) $\Leftrightarrow$ For every pair of hypersurfaces $M_1, M_2 \in \mathcal{F}$, any open pieces $U \subseteq M_1$ and $V \subseteq M_2$ are not CR diffeomorphic.

Applications

Recall: For $3-$dimensional strongly pseudoconvex hypersurfaces:

Bergman log. flatness $\Leftrightarrow$ Sphericity;

Obstruction flatness + Compactness + Transverse symmetry $\Rightarrow$ Sphericity.

Remark

Compact v.s. Noncompact

Proposition

(Ebenfelt-X., 2024) For every $n \geq 2$, $\exists$ an uncountably infinite family $\mathcal{F}$ of **(noncompact)** *real analytic CR hypersurfaces of dimension $2n+1$ such that*

(1) Every $M \in \mathcal{F}$ is obstruction flat and Bergman log. flat;

(2) Every $M \in \mathcal{F}$ is homogeneous and transversally symmetric;

(3) Local CR structures of hypersurfaces in $\mathcal{F}$ are mutually inequivalent.

Proof of Corollary:
Let's first use the special case of the main theorem where the base is a polydisk.

Let X be a compact Riemann surface of genus at least two (so that X is covered by the unit disk) and is equipped with the hyperbolic metric g.

Let (L, h) be the anti-canonical line bundle of (X, g).

Note: L is the holomorphic tangent vector bundle of X, and $h = g$.

Applications

Consider the product of n copies of X : $Y = X \times \cdots \times X$ (so that Y is covered by Δ^n).

Then consider the Hermitian line bundle over Y :

$$(\hat{L}, \hat{h}) := (L^{k_1}, h^{k_1}) \otimes \cdots \otimes (L^{k_n}, h^{k_n}).$$

where $(k_1, \cdots, k_n) \in \mathbb{N}^n$ is an $n-$tuple satisfying

$$1 = k_1 \leq k_2 \leq \cdots \leq k_n.$$

The circle bundle of $(\hat{L}, \hat{h})$ can be locally written as:

$$|\xi|^2 \prod_{i=1}^{n} \frac{1}{(1 - |z_i|^2)^{2k_i}} = 1.$$

Then via the local map $(z_1, \cdots, z_n, \xi) \to (z_1, \cdots, z_n, \sqrt{\xi})$, the above is further locally CR equivalent to

$$|\xi|^2 = \prod_{i=1}^{n}(1 - |z_i|^2)^{k_i}, |z_i| < 1 \text{ for } 1 \le i \le n.$$

Thus by varying the integers $1 = k_1 \le k_2 \le \cdots \le k_n$, we obtain the desired countable family of CR hypersurfaces.

Applications

Recall. Every bounded symmetric domain covers a compact complex manifold.

Remark. Instead of polydisk (= product of disks), we can use reducible bounded symmetric domains (= product of irreducible bounded symmetric domains).

In this way, we obtain a lot more flat CR hypersurfaces (which still form a family with mutually distinct local CR structures).

This larger family is, however, still countably infinite.

Thank you very much for your attention!