Deformation of Bergman Spaces

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Complex **A**nalysis, **Ge**ometry and **D**ynamics III

in
Portorož
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First example of deformation of Bergman spaces:

Fix Y_o Stein mfld, $X_o \subset\subset Y_o$ strictly ψcvx domain, $E_o \to Y_o$ holo v.b.

- Set $X = X_o \times B$ and $E = p_1^* E_o \to X$ for some cplx mfld B.
- Assume metric $\mathfrak{h}$ for E smooth up vertical boundary $\partial X_o \times B$.

To $(E, \mathfrak{h}) \to X \stackrel{p_2}{\to} B$ we associate

$$\mathscr{H}_t := \left\{ f \in H^0(X_o, \mathcal{O}(K_{X_o} \times E_o)) \ ; \ \int_X |f|^2_{\mathfrak{h}(\cdot,t)} < +\infty \right\}$$

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Note: $\mathscr{H}_t \subset H^0(X_o, \mathcal{O}(E_o))$ is independent of t.
Thus each $f \in \mathscr{H}_o$ defines 'constant' holo section of $\mathscr{H} \to B$.
$\Longrightarrow \mathscr{H} \to B$ is a trivial vector bundle.
But the L^2 metric is not 'constant' in t.

$\mathscr{H} \to B$ is a holomorphic vector bundle of infinite rank
Choose Hilbert basis $\{\phi_1, \phi_2, ...\} \subset \mathscr{H}_o$ for some $o \in B$.

- Sections $\mathfrak{f} \in \Gamma(B, \mathscr{H})$ correspond to

$$f(x,t) := \sum_j c_j(t)\phi_j(x) \quad \left(\text{such that } \int_{X_o} \left|\sum c_j(t)\phi_j\right|^2_{\mathfrak{h}(\cdot,t)} < +\infty\right)$$

- $\mathfrak{f}$ is smooth (resp. holo) iff each $c_j \in \mathscr{C}^\infty(B)$ (resp. $\mathcal{O}(B)$).

For a smooth section $\mathfrak{f}$, the standard operators

$$\begin{aligned}\bar{\partial}^{\mathscr{H}}_{\frac{\partial}{\partial \bar{t}^i}}\mathfrak{f} &\longleftrightarrow \frac{\partial f}{\partial \bar{t}^i} = \sum_j \frac{\partial c_j(t)}{\partial \bar{t}^i}\phi_j(x) \\ \nabla^{\mathscr{H}}_{\frac{\partial}{\partial t^i}}\mathfrak{f} &\longleftrightarrow \mathscr{P}\left(\nabla^E_{\frac{\partial}{\partial t^i}} f\right)(\cdot, t) \\ &\quad := \sum_i P_t\left(\frac{\partial c_j(t)}{\partial \bar{t}^i}\phi_j + c_j(t)\left(\nabla^{(E_o,\mathfrak{h}(\cdot,t))}\phi_j\right)\right)\end{aligned}$$

are only *densely defined*.
e.g., Domains contain all $\mathfrak{f}$ such that the corresponding section

$$f \in H^0(X, \mathscr{C}^\infty_X(K_{X/B} \otimes E))$$

is smooth up to the vertical boundary $\partial X_o \times B$.

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Theorem

Suppose that for each $t \in B$ the metric $\mathfrak{h}(\cdot, t)$ is Nakano-positive. If $(p^*E_o, \mathfrak{h}) \to X_o \times B$ is k-positive then so is $\mathscr{H} \to B$.

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- If E_o is a line bundle, this result is due to Berndtsson. In that case there is only one notion of positivity.
- Proof in the case of higher rank follows same lines

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- $\Theta(\mathfrak{h})$ uniquely defines a Hermitian form on $T^{1,0}_X \otimes E \to X$

$$\{\xi \otimes v, \eta \otimes w\}_{\mathfrak{h}} := \mathfrak{h}(\sqrt{-1}\Theta(\mathfrak{h})_{\xi\bar\eta} v, w)$$

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1-positive also called *Griffiths-positive*
k-positive $\forall k$ also called *Nakano-positive*

The L^2 bundle

Same trivial family $(p_1^* E_o, \mathfrak{h}) \to X_o \times B \xrightarrow{p_2} B$ with nontrivial metric defines $\mathscr{L} \to B$ where

$$\mathscr{L}_t := \left\{ f \in \Gamma(X_o, K_{X_o} \times E_o) \text{ msrable } ; \int_X |f|^2_{\mathfrak{h}(\cdot,t)} < +\infty \right\}, \ t \in B.$$

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  - ⋆ f is smooth if the a_j are smooth.
  - ⋆ As with $\mathscr{H} \to B$, derivatives might not be square-integrable.

Thus again we get densely defined operators

$$\frac{\partial \mathfrak{f}}{\partial \bar{t}^i} \longleftrightarrow \frac{\partial f(x,t)}{\partial \bar{t}^i} \quad \text{and} \quad \nabla^{\mathscr{L}}_{\frac{\partial}{\partial t^i}} \mathfrak{f} \longleftrightarrow \nabla^{E}_{\frac{\partial}{\partial t^i}} f.$$

Formula for the curvature: for $v, w \in T^{1,0}_{B,t} \subset T^{1,0}_X = T^{1,0}_{X_o} \oplus T^{1,0}_B$

$$(\Theta(\mathscr{L})_{v\bar{w}}\mathfrak{f}(t), \mathfrak{f}(t)) = \int_X \langle \Theta(\mathfrak{h}(\cdot,t))_{v\bar{w}} f, f \rangle_{\mathfrak{h}(\cdot,t)}.$$

Note:

- $\mathscr{H}_t$ is a subspace of $\mathscr{L}_t$ for all t,
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By Gauss-Griffiths Formula

$$(1) \quad (\Theta(\mathscr{L})f, f) = (\Theta(\mathscr{H})f, f) + (\mathbf{II}f, \mathbf{II}f) \quad \text{for all } f \in \Gamma(B, \mathscr{C}^{\infty}(\mathscr{H})),$$

where $\mathbf{II} : \mathscr{H} \to \mathscr{H}^{\perp}$ is the second fundamental form:

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$$(1)\quad (\Theta(\mathscr{L})f, f) = (\Theta(\mathscr{H})f, f) + (\mathbf{I\!I}f, \mathbf{I\!I}f) \quad \text{for all } f \in \Gamma(B, \mathscr{C}^{\infty}(\mathscr{H})),$$

where $\mathbf{I\!I} : \mathscr{H} \to \mathscr{H}^{\perp}$ is the second fundamental form:

$$\mathbf{I\!I}f := \nabla^{\mathscr{L}} f - \nabla^{\mathscr{H}} f = \nabla^{\mathscr{L}} f - P\nabla^{\mathscr{L}} f = P^{\perp}\nabla^{\mathscr{L}} f.$$

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Remark

Formula (1) agrees with our previous curvature formula

$$(\Theta(\mathscr{H})_{v\bar{w}}\mathfrak{f}, \mathfrak{g}) = \int_{X_o} \mathfrak{h}((\Theta(\mathfrak{h})_{v\bar{w}} f, g) - \int_{X_o} \mathfrak{h}\left(P_t^\perp \nabla_v f, P_t^\perp \nabla_w g\right).$$

Proper deformation of Bergman spaces

Take $(E, \mathfrak{h}) \to X \xrightarrow{p} B$ smooth proper (i.e., p is proper submersion).

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Definition (Holomorphic sections)

$$\mathfrak{f} \in \Gamma(B, \mathcal{O}_B(\mathscr{H})) \overset{\text{defn}}{\iff} f \in H^0(X, \mathcal{O}_X(K_{X/B} \otimes E))$$

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$$D(s) = \{0\}\times\mathbb{H} - \{s\}\times\mathbb{H} \quad \text{and}\ (z,t) \sim (\zeta, \tau) \iff t = \tau \ \&\ \ z - \zeta \in \mathbb{Z} \oplus t\mathbb{Z}$$

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- $D(s)|_{X_t} = \begin{cases} [0] - [s], & s \notin \mathbb{Z} \oplus t\mathbb{Z} \\ \mathcal{O} \quad , & s \in \mathbb{Z} \oplus t\mathbb{Z} \end{cases}$

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- X_t is the torus $\mathbb{C}/(\mathbb{Z} \oplus t\mathbb{Z})$,
- $D(s)|_{X_t} = \begin{cases} [0]-[s], & s \notin \mathbb{Z}\oplus t\mathbb{Z} \\ \mathcal{O}, & s \in \mathbb{Z}\oplus t\mathbb{Z} \end{cases}$
- $\dim_{\mathbb{C}} H^0(X_t, \mathcal{O}(K_{X_t} \otimes E(s)|_{X_t}) = \begin{cases} 0, & s \notin \mathbb{Z}\oplus t\mathbb{Z} \\ 1, & s \in \mathbb{Z}\oplus t\mathbb{Z} \end{cases}$

Let $(E, \mathfrak{h}) \to X \xrightarrow{p} B$ be proper, and assume X is Kähler.

Proposition (Berndtsson)

If $(E, \mathfrak{h}) \to X$ is Nakano non-negative then $\mathscr{H} \to B$ is a holo v.b.

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The L^2 extension theorem of Ohsawa-Takegoshi $\Rightarrow$
if $\mathfrak{h}$ is Nakano semi-positive then there is cts linear extension operator

$$\mathscr{E}_t : \mathscr{H}_t \to H^0(X, \mathcal{O}(K_X \otimes E)).$$

Thus $t \mapsto \dim \mathscr{H}_t$ is lower semi-continuous. □

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Remark

Note: $\bar{\partial}^{\mathscr{H}}$ is well-defined even if $\mathscr{H} \to B$ is not a v.b.

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On the other hand,

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\frac{\partial}{\partial t^j}(\mathfrak{f}, \mathfrak{g}) = \left(\nabla^{\mathscr{H}\, 1,0}_j \mathfrak{f}, \mathfrak{g} \right) + \left(\mathfrak{f}, \bar{\partial}^{\mathscr{H}}_j \mathfrak{g} \right).
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Therefore we have

$$\left(\nabla_j^{\mathscr{H}^{1,0}}\mathfrak{f}, \mathfrak{g}\right) = \int_{X_t} \left\langle P_t\left(L^{1,0}_{\xi_j} f(\cdot, t)\right) \wedge \overline{g(\cdot, t)}, \mathfrak{h}^t \right\rangle.$$

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To extract the formula

$$\nabla_j^{\mathscr{H}1,0}\mathfrak{f}\longleftrightarrow P_t\left(L^{1,0}_{\xi_j}f(\cdot,t)\right),$$

we need to know that

$$\text{(2)}\qquad \mathrm{eval}_t(H^0(B,\mathscr{C}^\infty(\mathscr{H})))\subset\mathscr{H}_t\quad\text{is dense for every } t\in B.$$

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Proposition

(2) *holds iff* $\mathscr{H} \to B$ *is a holo v.b.*

Therefore we have

$$\left(\nabla_j^{\mathscr{H}1,0}\mathfrak{f},\mathfrak{g}\right)=\int_{X_t}\left\langle P_t\left(L_{\xi_j}^{1,0}f(\cdot,t)\right)\wedge\overline{g(\cdot,t)},\mathfrak{h}^t\right\rangle.$$

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Proof.

As in previous proof:

Montel $\Rightarrow t\mapsto\dim\mathscr{H}_t$ u.s.c.

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Question

Can we extend such results to the non-locally trivial case?

The proper L^2 Hilbert field

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Because p is a holomorphic submersion, we get an isomorphism

$$T^{1,0}_{B,t} \ni v \overset{\cong}{\longleftrightarrow} \xi^\theta_v \in H^0(X_t, \mathscr{C}^\infty(T^{1,0}_X|_{X_t})) \text{ s.t. } 2\text{Re}\, \xi^\theta_v(x) \in \theta_x.$$

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Fact: $\bar\partial^{\mathscr{L}} \bar\partial^{\mathscr{L}} = 0 \iff [\theta^{1,0}, \theta^{1,0}] \subset \theta^{1,0}$
For most θ, $(\mathscr{L}, \bar\partial^{\mathscr{L}}) \to B$ is *not* a holo vector bundle (of infinite rank).

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Thus we have a Chern connection $\nabla = \nabla^{1,0} + \bar\partial$ (and its curvature).

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Proposition

If $\mathfrak{L}$ is a BLS field and $\mathfrak{H} \subset \mathfrak{L}$ is a BLS subfield then

a. $\nabla^{\mathfrak{H}} = P\nabla^{\mathfrak{L}}$, *and*
b. Gauss-Griffiths Formula: for all $\mathfrak{f}, \mathfrak{g} \in \mathscr{C}^\infty(\mathfrak{H})_t$ *and all* $v, w \in T^{1,0}_{B,t}$

$$(\Theta(\mathfrak{L})_{v\bar{w}}\mathfrak{f}, \mathfrak{g}) = (\Theta(\mathfrak{H})_{v\bar{w}}\mathfrak{f}, \mathfrak{g}) + (\mathbf{II}_v\mathfrak{f}, \mathbf{II}_w\mathfrak{g}),$$

where $\mathbf{II}\mathfrak{f} = \nabla^{\mathfrak{L}}\mathfrak{f} - \nabla^{\mathfrak{H}}\mathfrak{f} = P^\perp\nabla^{1,0}\mathfrak{f}$ *is the second fundamental form.*

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Theorem

The quantity

$$(\Theta(\mathscr{L})_{v\bar{w}}\mathfrak{f}, \mathfrak{g}) - (\mathbf{II}_v\mathfrak{f}, \mathbf{II}_w\mathfrak{g})$$

is independent of the horizontal distribution θ.

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Theorem (—)

There exists a projective manifold X, a Griffiths-positive holomorphic vector bundle $(E, \mathfrak{h}) \to X$ and a smooth complex hypersurface $Z \subset X$ such that the restriction map

$$H^0(X, \mathcal{O}_X(K_X \otimes E) \to H^0(Z, \mathcal{O}_Z((K_X \otimes E)|_Z))$$

is not surjective.

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Suppose $\mathscr{H} \to B$ *is Griffiths negative. Let* $\mathfrak{f} \in \Gamma(U, \mathscr{C}^{\infty}(\mathscr{L}))$ *such that* $\mathfrak{f}(t) \in \mathscr{H}_t$ *for all* $t \in U$*. Then* $\log(\mathfrak{f}, \mathfrak{f}) \in \mathrm{PSH}(U)$*.*

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Suppose $\mathscr{H}$ is k-positive. Let $t_o \in B$, let $U \subset B$ be a coordinate nbhd of t_o, and let $\mathfrak{f}_1, ..., \mathfrak{f}_k \in \Gamma(U, \mathscr{C}^\infty(\mathscr{L}))$ satisfy

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Then the (n,n)-form

$$-\sqrt{-1}\partial\bar{\partial} \sum_{i,j=1}^{k} (\mathfrak{f}_i, \mathfrak{f}_j)\, \Upsilon^{i\bar{j}}(z)$$

is positive at t_o, where $\Upsilon^{i\bar{j}}(z) := \dfrac{dV(z)}{\sqrt{-1}dz^i \wedge d\bar{z}^j}$.

Thanks for your attention.