ON THE KOBAYASHI METRICS IN RIEMANNIAN MANIFOLDS

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Abstract. We define the Kobayashi distance and the Kobayashi-Royden infinitesimal metric on any smooth Riemannian manifold (M, g), using conformal harmonic immersions from the unit disk in $\mathbb{C}$ into M. We also study their basic properties, following the approach developped by H.L.Royden [12] for complex manifolds.

1. Introduction

The Kobayashi metric and the related notion of hyperbolicity of complex manifolds play a fundamental role in the geometric Complex Analysis and in Algebraic Geometry. The definition of the Kobayashi metric crucially uses holomorphic discs i.e., the holomorphic maps from the unit disc $\mathbb{D}$ of $\mathbb{C}$ to a prescribed complex manifold. Of course, this definition requires the presence of a complex (or, at least, almost complex) structure on a manifold. In [4], M.Gromov proposed to extend the notion of Kobayashi metric to the case of arbitrary Riemannian manifolds. He suggested to use conformal harmonic maps or, equivalently, conformal maps whose images are minimal surfaces, from $\mathbb{D}$ to a Riemannian manifold (M, g). Recently, F.Forstnerič - D.Kalaj [3] and B.Drinovec-Drnovšek - F.Forstnerič [2] used this approach. They introduced the notion of Kobayashi metric and studied its important properties in the case of $\mathbb{R}^n$ equipped with the standard Euclidean metric g_{st}. The goal of the present paper is to consider the case of arbitrary Riemannian manifolds as it was suggested by M.Gromov. In particular, we extend to that general case some of the results of [2].

The paper is organized as follows. In Section 2 we recall some standard facts concerning minimal sufaces, conformal and harmonic maps. Lemma 2.3, giving an existence of a (small) minimal surface with prescribed center and tangent direction, is necessary for the definition of the Kobayashi pseudodistance and of the Kobayashi-Royden pseudometric. Note that our approach is based on important works of B.White [13, 14]. In Section 3, we introduce the Kobayashi-Royden pseudometric on a Riemannian manifold. The main result here is Theorem 3.3 establishing the upper semi-continuity of that pseudometric on the tangent bundle. In Section 4, we introduce the notion of Kobayashi pseudodistance on a Riemannian manifold. The main result of this paper is Theorem 4.1 which claims the coincidence of the

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Kobayashi pseudodistance with the integral form of the Kobayashi-Royden pseudometric. In the complex case, this is a classical theorem of H.L.Royden [12]. In the last Section 5, we discuss some basic properties of Riemannian manifolds which are hyperbolic in the sense of Kobayashi i.e., the associated Kobayashi pseudodistance is a distance.

Finally, we note that many open questions, such as the sufficient conditions for complete hyperbolicity, the existence of normal families of minimal surfaces, the asymptotic estimates of the Kobayashi-Royden metric near the boundary, or the Gromov hyperbolicity of the Kobayashi distance, stay open in the case of Riemannian manifolds. We will consider them in forthcoming papers. The authors are grateful to F. Forstnerič bringing their attention to the papers [2, 3].

2. Minimal immersions

We denote by $\mathbb{D}$ the unit disc in $\mathbb{R}^2$, by ds^2 the standard Riemannian metric on $\mathbb{R}^2$ and by dm the standard Lebesgue measure on $\mathbb{R}^2$. For every $x \in \mathbb{R}^2$ and every $\lambda > 0$, we set $D(x,\lambda) := \{\zeta \in \mathbb{C} | \ |\zeta| < \lambda\}$.

Let (M, g) be a Riemannian manifold. We assume that all structures are smooth of class C^∞. We denote by $dist_g$ the distance induced by g, defined as the infimum of the length of C^1 paths joining two points. For every $p \in M$ and every $r > 0$, let $\mathbb{B}(p, r) := \{q \in M | \ dist_g(p, q) < r\}$.

A map $u : \mathbb{D} \to M$ is called harmonic if it is a critical point of the energy integral

$$E(u) = \int_{\mathbb{D}} |du|^2 dm.$$

A harmonic map satisfies the Euler-Lagrange equations

$$\Delta_g u + B(u)(du, du) = 0. \tag{1}$$

Here Δ_g denotes the Laplace-Beltrami operator on (M, g). We do not develop the other notation used here (see for example [6]), since we will not need it. We only note that $u \mapsto \Delta_g u + B(u)(du, du)$ is a second order elliptic quasilinear PDE system. The initial regularity of u may be prescribed in the Hölder or Sobolev spaces. It follows by the elliptic regularity that u is a smooth C^∞ map on $\mathbb{D}$. Using the complex coordinates $z = x + iy$ on $\mathbb{D}$ and local coordinates on M near $u(\mathbb{D})$, one can rewrite the equations (1) in the form

$$\frac{\partial^2 u^i}{\partial z \partial \overline{z}} + \Gamma^i_{jk}(u(z)) \frac{\partial u^j}{\partial z} \frac{\partial u^k}{\partial \overline{z}} = 0 \tag{2}$$

(see [6]). Here $\Delta = 4\dfrac{\partial^2}{\partial z \partial \overline{z}}$ is the standard Laplace operator and Γ^i_{jk} denote the Christoffel symbols.

A smooth map $u : \mathbb{D} \to M$ is called conformal if the pull-back u^*g is a metric conformal to ds^2 i.e., there exists a smooth function ϕ such that $u^*g = e^\phi ds^2$ on $\mathbb{D}$. Recall that any Riemannian metric h on $\mathbb{D}$ admits conformal coordinates. This means that there exists a smooth diffeomorphism $\Phi : \mathbb{D} \to \mathbb{D}$, depending on h, such that Φ^*h is conformal to ds^2. This is a classical fact of differential geometry, see [6]. Note that this result is true without additional assumptions only for manifolds of dimension 2. If $u : \mathbb{D} \to M$ is a smooth immersion, we take $h = u^*g$, and the composition $u \circ \Phi$ becomes a conformal mapping. Of course, Φ depends on u and is not unique. Using the group of conformal automorphisms of $\mathbb{D}$, we can always achieve the conditions $\Phi(0) = 0$ and $\Phi(1) = 1$.

The condition for u to be conformal is equivalent to the conditions, satisfied for every $(x, y) \in \mathbb{D}$:

$$g_{u(x,y)}(\partial u/\partial x, \partial u/\partial x) = g_{u(x,y)}(\partial u/\partial y, \partial u/\partial y),\ g_{u(x,y)}(\partial u/\partial x, \partial u/\partial y) = 0 \tag{3}$$

(see [6]).

Recall that a surface in (M, g) is called minimal if its mean curvature (induced by g) vanishes. A conformal immersion (i.e., its image) is minimal if and only if it is harmonic (see [6]). The energy functional has very important compactness properties in suitable function spaces. This makes it a usuful tool in order to study boundary values problems for minimal surfaces, in particular, the Plateau problem. However, in some cases it is more convenient to work with the area functional. Here we follow the approach of B. White [13].

Let $u : \mathbb{D} \longrightarrow M$ be a smooth immersion. We denote by $g_{\mathbb{D}} := u^*(g)$ the Riemannian metric on $\mathbb{D}$, pullback of the metric g by u. Let $G_{\mathbb{D}}$ be the matrix $(G_{\mathbb{D}})_{i,j} = g_{\mathbb{D}}(\partial/\partial x_i, \partial/\partial x_j)$, for $i, j \in \{1, 2\}$ (for convenience, in the matrix notations $x_1 = x,\ x_2 = y$). We also denote $u_x := u_\star(\partial/\partial x)$ and $u_y := u_\star(\partial/\partial y)$. In particular, the scalar product of $\partial/\partial x_i, \partial/\partial x_j$ evaluated at $(x, y) \in \mathbb{D}$ is equal to $g_{\mathbb{D}}(\partial/\partial x_i, \partial/\partial x_j) = g_{u(x,y)}(u_{x_i}, u_{x_j})$. For convenience, we just write $g_{u(x,y)}(u_{x_i}, u_{x_j}) =: g_u(u_{x_i}, u_{x_j})$. Then the area functional $A(u)$ of the immersion u is defined by

$$A(u) = \int_{\mathbb{D}} (\det G_{\mathbb{D}})^{1/2} dm. \tag{4}$$

One may view A as a real map defined on the space of smooth immersions. A smooth immersion u is called **stationary** if the differential DA of A vanishes at u i.e., $DA(u) = 0$. As it is shown in [13], an immersion is stationary if and only if its image is a minimal surface. Therefore, after a suitable reparametrization, a stationary immersion becomes a conformal harmonic map.

In the rest of the paper, we refer conformal harmonic immersions from $\mathbb{D}$ to M as conformal harmonic immersed discs. Similarly, we refer to stationary or minimal discs. Note that we consider stationary discs with arbitrary parametrizations, not necessarily the conformal harmonic ones. Note also that we sometimes identify an immersed disc with its image when this does not lead to any confusion.

Since we work only locally, in suitable local coordinates on M we will consider $u : \mathbb{D} \to \mathbb{R}^n$ given as a graph :

$$\forall (x, y) \in \mathbb{D},\ u(x, y) = (x, y, u^3(x, y), \dots, u^n(x, y)),$$

where $u^3, \dots, u^n$ are smooth C^∞ functions. Then

$$u_x := (1, 0, u^3_x, \dots, u^n_x)$$

and

$$u_y := (1, 0, u^3_y, \dots, u^n_y).$$

Now we follow White's approach in [13, 14]. Let $h = (0, 0, h^3, \dots, h^n) : \mathbb{D} \to \mathbb{R}^n$ be a smooth map and let, for every $t \in [0, 1]$ and every $(x, y) \in \mathbb{D}$, $\varphi^t(x, y) = (x, y, u^3(x, y) +$

$th^3(x,y),\ldots,u^n(x,y)+th^n(x,y))$. Then

$$\varphi^t_x = (\varphi^t)_\star(\partial/\partial x) = (1,0,u^3_x + th^3_x,\ldots,u^n_x + th^n_x),$$

$$\varphi^t_y = (\varphi^t)_\star(\partial/\partial y) = (1,0,u^3_y + th^3_y,\ldots,u^n_y + th^n_y).$$

and

$$A(\varphi^t) = \int_{\mathbb{D}} \left(g_{\varphi^t}(\varphi^t_x,\varphi^t_x)g_{\varphi^t}(\varphi^t_y,\varphi^t_y) - g_{\varphi^t}(\varphi^t_x,\varphi^t_y)g_{\varphi^t}(\varphi^t_x,\varphi^t_y)\right)^{1/2} dxdy.$$

If we write $h_x = (0,0,h^3_x,\ldots,h^n_x)$ and $h_y = (0,0,h^3_y,\ldots,h^n_y)$, then we have, for every $t \in [0,1]$:

$$\varphi^0_x = u_x, \quad \frac{d}{dt}_{|t=0}\varphi^t_x = h_x$$

and

$$\varphi^0_y = u_y, \quad \frac{d}{dt}_{|t=0}\varphi^t_y = h_y.$$

Then u is a stationary immersion if and only if for every smooth map h we have

$$\frac{d}{dt}_{|t=0}A(\varphi^t) = 0.$$

The expression of $\frac{d}{dt}_{|t=0}A(\varphi^t)$ is straightforward and depends linearly on h, h_x and h_y. By integration by parts, we obtain a quasilinear operator, with respect to u_x, u_y, u_{xx} and u_{yy}, that depends linearly on h. Using the Riesz representation theorem (see details in [13]), we obtain the following

Lemma 2.1. *With these notations, the stationary condition has the form*

$$H(u) = 0 \tag{5}$$

where

$$H(u) = (H^1(u),...,H^n(u)) \tag{6}$$

with

$$H^j(u) = \psi^j(u_x,u_y,u_{xx},u_{xy},u_{yy}) \text{ for } j=1,2$$

and

$$H^j(u) = u^j_{xx} + u^j_{yy} + \psi^j(u_x,u_y,u_{xx},u_{xy},u_{yy}) \text{ for } j=3,\ldots,n.$$

Here, for $j=1,\ldots,n$, ψ^j is a smooth C^∞ function, without constant or linear terms with respect to $u_x,u_y,u_{xx},u_{xy},u_{yy}$. Furthermore, the vector $H(u)$ is orthogonal to $u(\mathbb{D})$.

In particular, the operator H is a quasilinear elliptic operator whose linearization at $u : (x,y) \in \mathbb{D} \mapsto (x,y,0,\ldots,0) \in \mathbb{R}^n$ is

$$\left(0,0,\Delta u^3,\cdots,\Delta u^n\right) \tag{7}$$

where Δ denotes the standard Laplace operator.

Since $H(u)$ is orthogonal to $u(\mathbb{D})$, the equation (5) is equivalent to the equations

$$H^j(u) = 0,\ j=3,...,n. \tag{8}$$

In what follows we mean these equations when refering to (5).

Example 2.2. *Consider the special case where* $M = \mathbb{R}^3$ *and* g *is the standard metric. Assume that the stationary map* $u : \mathbb{D} \to \mathbb{R}^3$ *is the graph of a function* $f : \mathbb{D} \to \mathbb{R}$ *i.e.,* $u : (x,y) \in \mathbb{D} \mapsto (x,y,f(x,y))$. *Then the equation (8) takes the form*

$$(1+f_y^2)f_{xx} + (1+f_x^2)f_{yy} - 2f_xf_yf_{xy} = 0. \tag{9}$$

Its linearization at $u : (x,y) \in \mathbb{D} \mapsto (x,y,0) \in \mathbb{R}^3$ *(i.e. the linearization of (9) at* $f = 0$*) is the standard Laplace operator* Δ.

We need the following classical regularity property of the Laplace operator. Consider the logarithmic potential

$$K(z) = (2\pi)^{-1} \log |z|. \tag{10}$$

This is a fundamental solution of the Laplace equation. Consider the Poisson equation

$$\Delta u = f\,. \tag{11}$$

Assume that f belongs to the Hölder class $C^\mu(\overline{\mathbb{D}})$, where μ is a positive real non-integer number. It is classical (see [10]) that the potential of f defined by

$$Uf(z) = \int_{\mathbb{D}} K(z-w)f(w)dm(w) \tag{12}$$

is a function of class $C^{2+\mu}(\overline{\mathbb{D}})$ and satisfies $\Delta U = f$. Furthermore, the linear operator $U : f \mapsto Uf$, $U : C^\mu(\overline{\mathbb{D}}) \to C^{2+\mu}(\overline{\mathbb{D}})$, is bounded.

We denote $e_1 := (1,0) \in \mathbb{R}^2$. The following result claims that locally there are many conformal harmonic maps.

Lemma 2.3.
(i) *Let* $p \in M$ *and* $\xi \in T_pM \setminus \{0\}$. *Then there exists a conformal harmonic immersion* $u : \mathbb{D} \to M$ *such that* $u(0) = p$ *and* $du(0) \cdot e_1 = \alpha\xi$ *for some* $\alpha > 0$. *Furthermore, this immersion depends smoothly on* p *and* ξ.
(ii) *Let* $u : \mathbb{D} \to M$ *be a conformal harmonic immersion. Suppose that* $u(\mathbb{D})$ *is contained in a ball* $\mathbb{B}(p,r)$ *of radius* $r > 0$ *small enough. Then there exists* $r_1 < r$ *and a smooth* $(n-2)$*-parameter foliation of* $\mathbb{B}(p,r_1)$ *by conformal harmonic discs containing* u *as a leaf. Moreover, given* q *close enough to* p, *and a vector* ξ *close enough to* $du(0) \cdot e_1$, *one can choose the above foliation such that for some leaf* $\tilde{u}$ *one has* $\tilde{u}(0) = q$ *and* $d\tilde{u}(0) \cdot e_1 = \xi$.

Proof. (i) We consider local coordinates $(x_1, \ldots, x_n)$ on M in which $p = 0$. We also assume that these coordinates are normal for the Levi - Civita connexion of (M,g). Then in these coordinates $g_{ij}(0) = \delta_{ij}$ and the first order partial derivatives of g_{ij} vanish at 0. Consider the metric $g_t(x_1, \ldots, x_n) := g(tx_1, \ldots, tx_n)$ for $t > 0$. We notice that such a metric is isometric to g. The metric $h_t = t^{-2}g_t$ is not isometric to g_t, but the corresponding set of stationary surfaces is the same as for g_t ; this follows immediately from the expression for the area functional and the definition of stationary immersions. Finally, note that the metric h_t converges to the standard metric g_{st} of $\mathbb{R}^n$ in any C^k-norm on any compact subset of $\mathbb{R}^n$, as $t \to 0$.

We may assume that $\xi = (1,0,...,0)$ and we search for a suitable stationary immersion of the unit disc of the form $(x,y) \mapsto (x,y,f(x,y))$ i.e., as the graph of a vector function

$f : \mathbb{D} \to \mathbb{R}^{n-2}$. Consider the equation (8) for the metric h_t. For $t = 0$, it becomes a vector analog of (9). Its linearization at $f = 0$ is given by (7) which is a surjective operator from $C^{2,\alpha}(\mathbb{D}, \mathbb{R}^{n-2})$ to $C^{0,\alpha}(\mathbb{D}, \mathbb{R}^{n-2})$, with any fixed $\alpha \in (0, 1)$. Indeed, this follows from the stated above regularity property of the logarighmic potential. By the implicit function theorem the equation (5) admits solutions for $t > 0$ small enough. Namely, for every sufficiently small t, there exists a minimal surface, given by a smooth stationary immersion $u_{t,p,\xi} : \mathbb{D} \to \mathbb{R}^n$, for the Riemannian metric h_t, such that $u_{t,p,\xi} : (x, y) \in \mathbb{D} \mapsto (x, y, f_{t,p,\xi}(x, y))$ depends smoothly on (t, p, ξ) and is a small deformation of the map $(x, y) \mapsto (x, y, 0)$. In particular, $u_{t,p,\xi}(0)$ is close to p and $du_{t,p,\xi}(0) \cdot e_1$ is close to ξ. Now, if $\mathcal{U}_p$ is a small neighborhood of p in $\mathbb{R}^n$ and $\mathcal{V}_\xi$ is a small neighborhood of ξ in the unit sphere (for the standard metric in $\mathbb{R}^n$), the same reasoning implies that for every sufficiently small t, the set $\{(u_{t,p',\xi'}(0), du_{t,p',\xi'}(0) \cdot e_1),\ p' \in \mathcal{U}_p,\ \xi' \in \mathcal{V}_\xi\}$ fills an open neighborhood of (p, ξ) in $\mathbb{R}^n \times \mathbb{R}^n$. Hence, for sufficiently small t, there exists $(p', \xi') \in \mathcal{U}_p \times \mathcal{V}_\xi$ such that $u_{t,p',\xi'}(0) = p$ and $du_{t,p',\xi'}(0) \cdot e_1 = c\xi$ for some real number c close to one. Note that all solutions are C^∞ smooth by the elliptic regularity. Finally, being a solution of the equation (5), $u_{t,p',\xi'} : \mathbb{D} \to \mathbb{R}^n$ is a stationary disc for g_t and therefore for g. Hence, this disc becomes a conformal harmonic immersion for g after a suitable reparametrization. The smooth dependence on p and ξ follows from the implicit function theorem. This proves Part (i) of Lemma 2.3 .

(ii) Choose local normal coordinates $(x_1, \dots, x_n)$ as above. Also, set $(\tilde{x}_1, \dots, \tilde{x}_n) = (tx_1, \dots, tx_n)$ for $t > 0$ small enough and again consider the metrics $h_t(\tilde{x}_1, \dots, \tilde{x}_n) = t^{-2} g(tx_1, \dots, tx_n)$. We may assume that, in the coordinates $(x_1, \dots, x_n)$, the conformal harmonic immersion $u : \mathbb{D} \to M$ has the form $u(x, y) = L(x, y) + O(|(x, y)|^2)$ where L is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^n$, of rank 2. Then the disc $u_t : (x, y) \in \mathbb{D} \mapsto u(tx, ty)$ has the expansion $u_t(x, y) = L(x, y) + O(t)$ in the coordinates $(\tilde{x}_1, \dots, \tilde{x}_n)$. As $t \to 0$, this family of discs converges to a conformal linear disc. Note that after an isometric (with respect to g_{st} structure) linear transformation this disc is a graph of the zero map. Then the desired result follows by the implicit function theorem as in part (i).

$\square$

Note that there is another way to prove Lemma 2.3, based on deep results on the Plateau problem. The following result is a very special case of the fundamental theorem of Morrey [11]. Let $\phi : \overline{\mathbb{D}} \to (M, g)$ be a smooth map. Suppose that a Riemannian manifold (M, g) satisfies some standard metric assumptions (for example, M is compact). Then there exists a conformal harmonic map $u : \mathbb{D} \to M$ continuous up to the boundary of the unit disc, and such that $u|_{b\mathbb{D}} = \phi|_{b\mathbb{D}}$. In particular, the image of u is a minimal surface in M with the boundary $u(b\mathbb{D})$. Furthermore, it follows from the results of Hildebrandt [7] that u depends continuously on the perturbation of ϕ. Choose local coordinates on M as above and consider all conformal linear maps from $\mathbb{D}$ to this local chart. Then the theorem of Morrey can be applied to the images of the unit circle by these maps. Since, in these local coordinates, the metric g is a small perturbation of the standard one, minimal surfaces given by Morrey's theorem are small deformations of the linear discs. This implies Lemma 2.3.

3. The Kobayashi-Royden pseudometric on a Riemannian manifold

Let (M, g) be a Riemannian manifold. For a point $p \in M$ and a tangent vector $\xi \in T_pM$ we set

$$F_M(p, \xi) := \inf \frac{1}{r}$$

where r runs over all positive real numbers for which there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ such that $u(0) = p$ and $du(0) \cdot e_1 = r\xi$. It follows by Lemma 2.3 that F_M is well defined for every (p, ξ) in the tangent bundle TM. We call F_M the *Kobayashi-Royden pseudometric* for the Riemannian manifold (M, g).

We have the following obvious

Lemma 3.1. *Let $f : (M, g) \to (N, h)$ be an isometric immersion between two Riemannian manifolds i.e., f satisfies $g = f^*h$. Then*

$$F_N(f(p), df(p)\xi) \leq F_M(p, \xi).$$

In particular, if M is a connected open subset of N, then

$$F_N(p, \xi) \leq F_M(p, \xi).$$

For $(p, \xi) \in TM$, we denote $|\xi|_g := \sqrt{g_p(\xi, \xi)}$, when no confusion is possible.

Lemma 3.2. *The function F_M is non-negative, and for any real a one has*

$$F_M(p, a\xi) = |a| F_M(p, \xi).$$

If K is a compact subset of M then there is a constant $C_K > 0$ such that for $p \in K$ one has

$$F_M(p, \xi) \leq C_K |\xi|_g.$$

Proof. The first property is obvious. Let us prove the second one. For every point $p \in M$ and every sufficiently small open neighborhood U of p, Lemma 2.3 implies that there exists $\varepsilon > 0$ with the following property: for every point $q \in U$, every unit vector $\xi \in T_qM$ and every $r \in (0, \varepsilon)$, there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ such that $u(0) = q$, $du(0) \cdot e_1 = r\xi$. Then for $C_U = 1/\varepsilon$ we have $F(q, \xi) \leq C_U |\xi|_g$ for all vectors ξ (not nesessarily unit). Covering K by a finite number of open neighborhoods, we conclude. $\square$

The main regularity property is given by the following

Theorem 3.3. *The function F_M is upper semi-continuous on the tangent bundle TM.*

The key result needed for the proof is the following

Proposition 3.4. *Let $u_0 : \mathbb{D} \to M$ be a conformal harmonic immersion. Then there exists a neighborhood U of $(u_0(0), du_0(0) \cdot e_1)$ in the tangent bundle TM such that for each $(p, \xi) \in U$ there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ satisfying $u(0) = p$, $du(0) \cdot e_1 = \xi$.*

The upper semicontinuity of F_M follows immediately from this proposition and the definition of F_M. So the remainder of this section is devoted to the proof of Proposition 3.4.

Proof. Essentially, the proof is implicitely contained in the works [13, 14]. For the convenience of the reader we include some details. For simplicity we consider the case whee M coincides with $\mathbb{R}^n$ equipped with a Riemannian metric g. It is indicated in [13, 14] how the theory of White can be extended to arbitrary ambient Riemannian manifolds.

The operator $DH(u_0)$ is called the Jacobi operator at u_0. In order words, the Jacobi operator is the linearization of H at u_0. This is a second order linear PDE operator. Furthermore, it is elliptic and self-adjoint. If a vector field h is tangent to the disc $u_0(\mathbb{D})$, then the value of $DH(u_0)$ on h, i.e., $DH(u_0)\cdot h$, vanishes. Thus, for any h, the vector field $DH(u_0)\cdot h$ is normal to $u_0(\mathbb{D})$. Similarly to Section 2 (cf. the equations (5) and (8)), in what follows we identify $DH(u_0)$ with its normal projection (with respect to $u_0(\mathbb{D})$). A Jacobi field with respect to u_0 is a vector field in the kernel of $DH(u_0)$. It is established in [13] (Corollary at Section 6) that a smooth Jacobi field at u_0 is the initial velocity vector field of a one-parameter family of stationary discs. In order to apply this result, we need to study the existence of Jacobi fields.

The existence of the Jacobi fields in the direction transverse to $u_0(\mathbb{D})$ is established in [14]. Consider a sufficiently small $r > 0$ and the disc $u_0(r\mathbb{D})$. By Lemma 2.3 this disc generates a one-parameter family of minimal discs in any direction tranverse to $u_0(\mathbb{D})$ at the origin. Such a deformation is generated by a Jacobi field, obtained by taking the derivative with respect to the parameter, defined in a neighborhood of the origin on $u_0(\mathbb{D})$: this provides the existence of a Jacobi field with respect to u_0 in a prescribed direction near the centre $u_0(0)$. The existence of a global Jacobi field now follows from P.Lax' Equivalence Theorem of [9],stating, in our case, that a Jacobi field defined in a neighborhood of 0 in $\mathbb{R}^2$ can be approximated by a Jacobi field defined on the whole disc $\mathbb{D}$. The property of unique continuation of solutions required by theorem of P.Lax follows in our case from the Calderon uniqueness theorem, see [1], Theorem 11. Now Proposition 3.4 follows. □

4. The Kobayashi distance and coincidence theorem

Denote by $P_{\mathbb{D}}$ the Poincaré metric, defined for $z \in \mathbb{D}$ and $v \in \mathbb{C}$ by

$$P_{\mathbb{D}}(z, v) = \frac{|v|}{1 - |z|^2}$$

and by $\rho_{\mathbb{D}}$ the Poincaré distance on $\mathbb{D}$. Recall that for every $z, w \in \mathbb{D}$:

$$\rho_{\mathbb{D}}(z, w) = \frac{1}{2} \log \frac{1 + d(z, w)}{1 - d(z, w)}$$

where

$$d(z, w) = \frac{|z - w|}{|1 - z\overline{w}|}.$$

Let p and q be two points in the Riemannian manifold (M, g). A Kobayashi chain from p to q is a finite sequence of points z_k, w_k in $\mathbb{D}$ and of conformal harmonic immersions $u_k : \mathbb{D} \to M$, $k = 1, \dots, m$, such that $u_1(z_1) = p$, $u_m(w_m) = q$ and $u_k(w_k) = u_{k+1}(z_{k+1})$ for $k = 1, \dots, m - 1$. The existence of such chains follows from Lemma 2.3. Indeed, by compactness any smooth path from p to q can be covered by sufficiently small balls such that the centre of a given ball is contained in the preceding ball. By Lemma 2.3 each ball is foliated by minimal discs through its centre, which provides a Kobayashi chain. The Kobayashi pseudodistance from p to q is then defined by

$$d_M(p, q) = \inf \sum_k \rho_{\mathbb{D}}(z_k, w_k) \tag{13}$$

where the infimum is taken over all Kobayashi chains from p to q.

On another hand, we consider the pseudodistance defined as the integrated form of F_M:

$$\overline{d}_M(p,q) = \inf \int_0^1 F_M(\gamma(t), \dot\gamma(t))dt \tag{14}$$

where the infimum is taken over all piecewise smooth paths γ from p to q. Notice that $\overline{d}_M$ is well defined by Theorem 3.3. The main result of this section is the following coincidence theorem.

Theorem 4.1. *We have $d_M = \overline{d}_M$.*

Proof. First we note that for every conformal harmonic disc $u : \mathbb{D} \to M$ we have, for every $z \in \mathbb{D}$ and every $\xi \in \mathbb{C}$:

$$F_M(u(z), du(z)\cdot\xi) \le P_{\mathbb{D}}(z,\xi). \tag{15}$$

Indeed, in the case where $z = 0$ and $\xi = e_1$, this follows from the definition of F_M. Then it follows for any $z \in \mathbb{D}$ and any unit vector ξ, if we replace u with the conformal harmonic disc $u\circ\phi$, where ϕ is a biholomorphic automorphism of $\mathbb{D}$ such that $\phi(0) = z$, $d\phi(0)\cdot e_1 = \xi$. Then the inequality (15) follows for any vector ξ since the two metrics are absolutely homogeneous. As a consequence, for any conformal harmonic disc $u : \mathbb{D} \to M$ we have for every $z, w \in \mathbb{D}$:

$$\overline{d}_M(u(z), u(w)) \le \rho_{\mathbb{D}}(z,w). \tag{16}$$

Consider now a Kobayashi chain between p and q, as above. Then, by the triangle inequality and the inequality (16) we have:

$$\overline{d}_M(p,q) \le \sum_k \overline{d}_M(u_k(z_k), u_k(w_k)) \le \sum \rho_{\mathbb{D}}(z_k, w_k).$$

Taking the infimum over all Kobayashi chains joining p to q, we obtain that $\overline{d}_M(p,q) \le d_M(p,q)$.

Let us prove now the converse inequality. Fix $\varepsilon > 0$ and consider a smooth path $\gamma : [0,1] \to M$ satisfying $\gamma(0) = p$, $\gamma(1) = q$ and such that

$$\int_0^1 F_M(\gamma(t),\dot\gamma(t))dt < \overline{d}_M(p,q) + \varepsilon.$$

Since F_M is an upper-semicontinuous function, there is a continuous function $h : [0,1] \to \mathbb{R}$ satisfying $h(t) > F_M(\gamma(t),\dot\gamma(t))$ for every $t \in [0,1]$ and such that

$$\int_0^1 h(t)dt < \overline{d}_M(p,q) + \varepsilon.$$

Therefore, for every sufficiently fine partition $0 = t_0 < t_1 < ... < t_m = 1$ of $[0,1]$ we have

$$\sum_{i=1}^m h(t_{i-1})(t_i - t_{i-1}) < \overline{d}_M(p,q) + \varepsilon.$$

Lemma 4.2. *There exist constants $C > 0$ and $\delta > 0$ such that for every $t \in [0,1]$ and every $a, b \in \mathbb{B}(\gamma(t),\delta)$ we have:*

$$d_M(a,b) \le C dist_g(a,b)$$

where $dist_g$ denotes the distance induced by the Riemannian metric g on M.

Proof. Fix $t \in [0,1]$. Choose local normal coordinates centered at the point $\gamma(t)$ so that this point is the origin in these coordinates. By Lemma 2.3 there exists $\delta_1 > 0$ small enough such that for any vector $\xi \in T_{\gamma(t)}M \setminus \{0\}$, the ball $\mathbb{B}(\gamma(t), \delta_1)$ contains an immersed minimal disc centered at $\gamma(t)$ and tangent to the direction of ξ. It is shown in the proof of Lemma 2.3 that the family D of these discs is a small smooth perturbation of the family of linear conformal discs which fill the ball $\mathbb{B}(\gamma(t), \delta_1)$. Hence, shrinking δ_1 if necessary, we may assume that for every point $b \in \mathbb{B}(\gamma(t), \delta_1)$ there exists a disc from the family D, centered at $\gamma(t)$ and passing through b. It also follows from the proof of Lemma 2.3 (which is based on an application of the implicit function theorem) that the above family D of minimal discs smoothly depends on a small perturbation of their common center $\gamma(t)$.

Hence there exist $0 < \delta < \delta_1/2$ and, for each point $a \in \mathbb{B}(\gamma(t), \delta)$, a family D_a of minimal discs centered at a such that the family D_a still fills the ball $\mathbb{B}(\gamma(t), \delta_1/2)$. In other words for any a, b in $\mathbb{B}(\gamma(t), \delta)$ there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ through a and b. The map u being an immersion, there exists $C > 0$, only depending on δ, such that

$$dist_g(a, b) = dist_g(u(z), u(w)) \geq C|z - w|.$$

Moreover, by changing δ is necessary, there exists $C' > 0$, only depending on δ, such that $\rho_{\mathbb{D}}(z, w) \leq C'|w - z|$.

Now it follows from the definition of the Kobayashi distance that

$$d_M(a, b) \leq \rho_{\mathbb{D}}(z, w) \leq C|w - z| \leq C|b - a|.$$

(Here, the constant $C > 0$ changes from inequality to inequality.)

All constants being uniform, this proves Lemma 4.2. □

Fix $t \in [0,1]$. Since $h(t) > F_M(\gamma(t), \dot\gamma(t))$, there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ and a real number $r > 0$ such that $u(0) = \gamma(t)$, $du(0) \cdot e_1 = r\dot\gamma(t)$ and $h(t) > 1/r$. Therefore, for every s real close enough to the origin we have the expansion, in local coordinates in M:

$$u(s/r) = \gamma(t) + (s/r)du(0) \cdot e_1 + O(s^2) = \gamma(t) + s\dot\gamma(t) + O(s^2).$$

Note also that $\rho_{\mathbb{D}}(0, z) = |z| + O(|z|^2)$ for $z \in \mathbb{D}$ close enough to 0. Therefore, using Lemma 4.2 we obtain:

$$\begin{aligned} d_M(\gamma(t), \gamma(t) + s\dot\gamma(t)) &\leq d_M(u(0), u(s/r)) + d_M(u(s/r), \gamma(t) + s\dot\gamma(t)) \\ &\leq \rho_{\mathbb{D}}(0, s/r) + C dist_g(u(s/r), \gamma(t) + s\dot\gamma(t)) \\ &\leq |s|/r + O(s^2) \\ &< |s|h(t) + O(s^2). \end{aligned}$$

Using again Lemma 4.2, we conclude that for $t, \tilde t \in [0,1]$, close enough, one has

$$\begin{aligned} d_M(\gamma(t), \gamma(\tilde t)) &\leq d_M(\gamma(t), \gamma(t) + (\tilde t - t)\dot\gamma(t)) + d_M(\gamma(t) + (\tilde t - t)\dot\gamma(t), \gamma(\tilde t)) \\ &\leq |\tilde t - t|h(t) + O(|\tilde t - t|^2). \end{aligned}$$

As a consequence, for any sufficiently fine partition we have

$$d_M(p, q) \leq \sum_{i=1}^{m} d_M(\gamma(t_{i-1}), \gamma(t_i)) \leq \sum_{i=1}^{m} (t_i - t_{i-1})h(t_{i-1})(1 + \varepsilon) < (1 + \varepsilon)(\overline{d}_M(p, q) + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, the proof is concluded.

$\square$

5. Hyperbolicity

The main purpose of this Section is to prove an analogue of [12], Theorem 2 p. 133, in the context of Riemannian manifolds. In case $(M, g) = (\mathbb{R}^n, g_{st})$, this is proved in [2], Theorem 4.2. We follow the presentation of [12].

The notion of hyperbolicity we consider in Definition 5.1 (ii) comes from complex geometry and refers to the Kobayashi-Royden pseudometric defined in Section 3. It is a local notion, defined at every point, and is different from the classical concept of a hyperbolic complete Riemannian manifold, that has constant sectional curvature equal to -1.

Definition 5.1. (i) *A Riemannian manifold* (M, g) *is called* Kobayashi hyperbolic *if the Kobayashi pseudodistance* d_M *is a distance.*

(ii) *A Riemannian manifold* (M, g) *is called* hyperbolic *at a point* $x \in M$ *if there is a neighborhood* U *of* x *and a positive constant* c *such that* $F_M(y, \xi) \geq c|\xi|_g$ *for every* $y \in U$ *and every* $\xi \in T_yM$*.*

(iii) *A Riemannian manifold* (M, g) *is called* tight *if the family of conformal harmonic immersions from* $\mathbb{D}$ *to* M *is equicontinuous for the topology generated by* $dist_g$*.*

(iv) *A family* $\mathcal{F}$ *of mappings of a topological space* X *into a topological space* Y *is called* even *if, given* $x \in X$, $y \in Y$ *and a neighborhood* U *of* y*, there is a neighborhood* V *of* x *and a neighborhood* W *of* y *such that for every* $f \in F$*, we have* $f_{|V} \subset U$ *whenever* $f(x) \in W$*.*

We have the following characterization of hyperbolicity of Riemannian manifolds

Theorem 5.2. *Let* (M, g) *be a Riemannian manifold. Then the following statements are equivalent:*

(i) *the family* $\mathcal{CH}(\mathbb{D}, M)$ *of conformal harmonic immersions from* $\mathbb{D}$ *to* M *is equicontinuous with respect to the distance* $dist_g$ *i.e.,* $(M, dist_g)$ *is tight,*

(ii) *the family* $\mathcal{CH}(\mathbb{D}, M)$ *is an even family,*

(iii) M *is hyperbolic at every point,*

(iv) d_M *is a distance i.e.,* M *is Kobayashi hyperbolic,*

(v) *the Kobayashi metric* d_M *induces the usual topology of* M*.*

Proof. The implication $(i) \Rightarrow (ii)$ follows from [8] p. 237, where it is stated that equicontinuity of a family of mappings of a topological space X into a topological space Y with respect to any metric inducing the topology of Y implies that the family is an even family.

$(ii) \Rightarrow (iii)$. This is based on the following result, see [5], Theorem 4.8.1 p. 113 :

Theorem (Schwarz Lemma). *Let* (X, h) *and* (Y, g) *be Riemannian manifolds. Let* $\mathbb{B}(x_0, R_0) \subset X$, $R_0 < \min\left(i_X(x_0), \frac{\pi}{2\kappa_X}\right)$*, where* $i_X(x_0)$ *denotes the injectivity radius at* x_0 *and* $-\omega_X^2 \leq K_X \leq \kappa_X^2$ *are curvature bounds on* $\mathbb{B}(x_0, R_0)$*. Let* $\mathbb{B}(y_0, R') \subset Y$, $R' < \min\left(i_Y(y_0), \frac{\pi}{2\kappa_Y}\right)$*, where* $i_Y(y_0)$ *denotes the injectivity radius at* y_0 *and* $-\omega_Y^2 \leq K_Y \leq \kappa_Y^2$ *are curvature bounds on* $\mathbb{B}(y_0, R')$*.*

If $u : X \to \mathbb{B}(y_0, R')$ *is harmonic, then for all* $R \leq R_0$*:*

$$|\nabla u(x_0)| \leq c_0 \max_{x \in B(x_0,R)} \frac{dist_g(u(x), u(x_0))}{R} \tag{17}$$

where $c_0 = c_0(R_0, \omega_X, \kappa_X, \dim X, R', \omega_Y, \kappa_Y, \dim Y)$*.*

Let p be a point in M and let $\mathbb{B}(p, R')$ such that R' satisfies the condition of the above Schwarz Lemma, with $y_0 = p$. By assumption, the family $\mathcal{CH}(\mathbb{D}, M)$ being even, there exist $0 < \delta < 1$ and $0 < \delta' < R'$, such that for every conformal harmonic immersion $u : \mathbb{D} \to M$, we have $u_{|D(0,\delta)} \subset B(p, R')$, whenever $u(0) \in B(p, \delta')$. It follows by the Schwarz Lemma that there exists $c_0 > 0$ such that for every conformal harmonic immersion $u : \mathbb{D} \to M$, with $u(0) \in \mathbb{B}(p, \delta')$, we have

$$|\nabla u(0)| \leq c_0 \frac{R'}{\delta}.$$

Hence, for every $y \in \mathbb{B}(p, \delta')$ and for every $v \in T_y M$, we have: $F_M(y, v) \geq \dfrac{\delta}{c_0 R'} |v|_g$.

$(iii) \Rightarrow (iv)$. By the assumption on F_M and the definition of $\overline{d}_M$, $\overline{d}_M$ is a distance. The implication is then given by Theorem 4.1.

$(iv) \Rightarrow (v)$. Let $x \in M$ and let $\delta > 0$ be such that $\mathbb{B}(x, \delta)$ is geodesically convex i.e., for every $y, z \in \mathbb{B}(x, \delta)$, there exists $\gamma_0 : [0, 1] \to M$ a piecewise C^1 smooth path joining y to z, with $\gamma_0([0, 1]) \subset \mathbb{B}(x, \delta)$ and $l_g(\gamma_0) := \int_0^1 |\dot{\gamma_0}(t)|_g dt = dist_g(y, z)$. Applying Lemma 3.2 to the compact set $K := \overline{\mathbb{B}(x, \delta)}$, there exists $C_K > 0$, such that $F_M(y, v) \leq C_K |v|_g$, for every $y \in K$ and every $v \in T_y M$. It follows now from Theorem 4.1:

$$d_M(y, z) = \overline{d}_M(y, z) \leq \int_0^1 F_M(\gamma_0(t), \dot{\gamma_0}(t)) dt \leq C_K \int_0^1 |\dot{\gamma_0}(t)|_g dt = C_K dist_g(y, z). \tag{18}$$

This means that the topology generated by d_M is weaker than the topology generated by $dist_g$.

Moreover, it follows from (18) that the map $y \mapsto d_M(x, y)$ is continuous on $\mathbb{B}(x, \delta)$ for the topology generated by $dist_g$. In particular, for every $0 < \alpha < \delta$, there exists $y_0 \in \partial \mathbb{B}(x, \alpha/2)$ such that

$$d_M(x, y_0) = \inf_{y \in \partial \mathbb{B}(x, \alpha/2)} d(x, y) > 0.$$

By the definition of $\overline{d}_M$, for every $y \in M \setminus \overline{\mathbb{B}(x, \alpha)}$

$$d_M(x, y) = \overline{d}_M(x, y) \geq \overline{d}_M(x, y_0) = d_M(x, y_0).$$

This means that the set $\{y \in M / d_M(x, y) < d_M(x, y_0)\} \subset \mathbb{B}(x, \alpha)$ i.e., the topology generated by $dist_g$ is weaker than the topology generated by d_M.

$(v) \Rightarrow (i)$. It follows from (13) et (14) that every conformal harmonic immersion $f : \mathbb{D} \to M$ and for every $\zeta, \zeta' \in \mathbb{D}$: $d_M(f(\zeta), f(\zeta')) \leq \rho_{\mathbb{D}}(\zeta, \zeta')$. Hence the family $\mathcal{CH}(\mathbb{D}, M)$ is equicontinuous. □

We conclude by noticing that, as in the complex setting and following [12], we can define the notions of tautness and complete hyperbolicity for Riemannian manifolds :

Definition 5.3. (i) *A Riemannian manifold (M,g) is called taut if the family $\mathcal{CH}(\mathbb{D},M)$ is a normal family i.e., every sequence in $\mathcal{CH}(\mathbb{D},M)$ admits a subsequence that either converges to an element of $\mathcal{CH}(\mathbb{D},M)$ uniformly on compact subsets of $\mathbb{D}$, or is compactly divergent.*

(ii) *A Riemannian manifold (M,g) is called complete hyperbolic if it is Kobayashi hyperbolic and the metric space (M,d_M) is complete.*

Then the characterization of tautness and complete hyperbolicity contained in Section 4 of [12] still hold in our context. In particular we have the following

Proposition 5.4. *Let (M,g) be a Riemannian manifold. Then we have:*

(i) *M is complete hyperbolic if and only if for every $p \in M$ and every $r>0$, the set $\{q \in M/\ d_M(p,q) \leq r\}$ is compact in M.*

(ii) *M complete hyperbolic $\Rightarrow$ M taut $\Rightarrow$ M Kobayashi hyperbolic.*

(ii) *Let $\pi : \tilde{M} \to M$ be a covering map of M and denote by $\tilde{g}$ the unique Riemannian metric on $\tilde{M}$ such that π is a Riemannian covering map. Then $(\tilde{M},\tilde{g})$ is complete hyperbolic if and only if (M,g) is complete hyperbolic.*

We denote $e_1 := (1,0) \in \mathbb{R}^2$. The following result claims the existence of a conformal harmonic map with prescribed 1-jet.

Lemma 5.5. *Let $p \in M$ and let E be a 2-dimensional subspace of T_pM. Then there exists a conformal harmonic immersion $u : \mathbb{D} \to M$ with $u(0) = p$ and such that the tangent space $T_pu(\Delta)$ conicides with E. Furthermore, this immersion depends smoothly on p and E.*

Proof. We consider local coordinates $(x_1,\dots,x_n)$ on M in which $p=0$. We also assume that these coordinates are normal for the Levi - Civita connexion of (M,g). Then in these coordinates $g_{ij}(0) = \delta_{ij}$ and the first order partial derivatives of g_{ij} vanish at 0. Consider the metric $g_t(x_1,\dots,x_n) := g(tx_1,\dots,tx_n)$ for $t>0$. We notice that such a metric is isometric to g. The metric $h_t = t^{-2}g_t$ is not isometric to g_t, but the corresponding set of stationary surfaces is the same as for g_t ; this follows immediately from the expression for the area functional and the definition of stationary immersions. Finally, note that the metric h_t converges to the standard metric g_{st} of $\mathbb{R}^n$ in any C^k-norm on any compact subset of $\mathbb{R}^n$, as $t \to 0$.

We may assume that E is generated by the vectors $(1,0,...,0)$ and $(0,1,...,0)$. We search for a suitable stationary immersion of the unit disc of the form $(x,y) \mapsto (x,y,f(x,y))$ i.e., as the graph of a vector function $f : \mathbb{D} \to \mathbb{R}^{n-2}$. Consider the equation (8) for the metric h_t. Its linearization at $f=0$ is given by (7) which is a surjective operator from $C^{2,\alpha}(\mathbb{D},\mathbb{R}^{n-2})$ to $C^{0,\alpha}(\mathbb{D},\mathbb{R}^{n-2})$, with any fixed $\alpha \in (0,1)$. Indeed, this follows from the stated above regularity property of the Laplace operator in the Holder scale. By the implicit function theorem the equation (5) admits solutions for $t>0$ small enough. Namely, for every sufficiently small t, there exists a minimal surface, given by a smooth stationary immersion $u(t,p,E) : \mathbb{D} \to \mathbb{R}^n$, for the Riemannian metric h_t, such that $u(t,p,E) : (x,y) \in \mathbb{D} \mapsto (x,y,f_{t,p,E}(x,y))$ depends smoothly on (t,p,E) and is a small deformation of the map $(x,y) \mapsto (x,y,0)$. In particular, $u(t,p,E)(0)$ is close to p and $du(t,p,E)(0)(T_0\mathbb{D})$ is close to E. Now, if $\mathcal{U}_p$ is a small neighborhood of p in $\mathbb{R}^n$ and $\mathcal{V}$ is a small neighborhood of E in the Grassmanian $Gr(2,n)$, the same reasoning implies that for every sufficiently small t, the set $(u(t,p,E)(0), du(t,p,E)(0)(T_0\mathbb{D}))$ where $p' \in \mathcal{U}_p$ and $E \in \mathcal{V}$, fills an open neighborhood of (p,E) in $\mathbb{R}^n \times Gr(2,n)$. Hence, for sufficiently small t, there exists $(p,E) \in \mathcal{U}_p \times \mathcal{V}$ such that $u(t,p,E)(0) = p$ and $du(t,p,E)(0)(T_0\mathbb{D}) = E$. Note that all solutions are C^∞ smooth by the elliptic regularity. Finally, being a solution of the equation (5), $u(t,p,E) : \mathbb{D} \to \mathbb{R}^n$ is a stationary disc for g_t and therefore for g. Hence, this disc becomes a conformal harmonic immersion for g after a suitable reparametrization. The smooth dependence on p and E follows from the implicit function theorem. This proves Lemma 5.5 . □

5.1. **MPSH functions.** Let ρ be a function of class C^2 on M.

In local coordinates at a point $p \in M$, the Hessian of ρ is given by

$$\nabla^2\rho = \sum_{i,j=1}^n \left(\frac{\partial^2\rho}{\partial x_i \partial x_j} - \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial \rho}{\partial x_k} \right) dx_i \otimes dx_j. \tag{19}$$

Following [2], we have the following

Definition 5.6. *Let $\rho : M \to [-\infty, +\infty)$ be an upper semi-continuous function. We say that ρ is minimal plurisubharmonic (MPSH) if for every conformal harmonic immersed disc $u : \mathbb{D} \to M$, the composition $\rho \circ u$ is subharmonic on $\mathbb{D}$.*

In the case of C^2 functions, we have the following characterization of minimal plurisubharmonicity

Lemma 5.7. *Let $\rho : M \to \mathbb{R}$ be a function of class C^2. Then ρ is MPSH if and only if $\Delta(\rho \circ u) \geq 0$ for every conformal harmonic immersed disc $u : \mathbb{D} \to M$.*

Since a conformal harmonic disc has only a finite number of singular points on compact subsets of $\mathbb{D}$, then a function $\rho : M \to \mathbb{R}$ of class C^2 is MPSH if and only if $\Delta(\rho \circ u) \geq 0$ for every conformal harmonic disc $u : \mathbb{D} \to M$. Moreover, there are sufficiently many immersed harmonic discs for the MPSH notion to be consistent, as showed by Lemma 5.5.

Lemma 5.8. *Let ρ be a C^2 function on M and $u : \mathbb{D} \to M$ be a conformal harmonic map. Then $\Delta(\rho \circ u)(0)$ depends only on the 1-jet of u at the origin.*

Proof. We set $\partial/\partial x = (1,0)$, $\partial/\partial y = (0,1)$, $p = u(0)$. Let $(x_1, \dots, x_n)$ be normal coordinates at p and let $u = (u_1, \dots, u_n)$. We recall that $\Gamma^j_{k,l}(p) = 0$, where $\Gamma^j_{k,l}$ denote the Christoffel symbols of u in the normal coordinates. Then :

$$\frac{\partial^2(\rho \circ u)}{\partial x^2}(0) = \nabla^2\rho(p) \left(\sum_{j=1}^n \frac{\partial u_j}{\partial x}(0), \sum_{k=1}^n \frac{\partial u_k}{\partial x}(0) \right) + \nabla\rho(p) \left(\sum_{j=1}^n \frac{\partial^2 u_j}{\partial x^2}(0) \right)$$

and

$$\frac{\partial^2(\rho \circ u)}{\partial y^2}(0) = \nabla^2\rho(p) \left(\sum_{j=1}^n \frac{\partial u_j}{\partial y}(0), \sum_{k=1}^n \frac{\partial u_k}{\partial y}(0) \right) + \nabla\rho(p) \left(\sum_{j=1}^n \frac{\partial^2 u_j}{\partial y^2}(0) \right).$$

Since u is harmonic, it follows from (**??**) that

$$\sum_{j=1}^n \left(\frac{\partial^2 u_j}{\partial x^2}(0) + \frac{\partial^2 u_j}{\partial y^2}(0) \right) = - \sum_{j,k,l=1}^n \Gamma^j_{k,l}(p) \frac{\partial u_k}{\partial x}(0) \frac{\partial u_l}{\partial x}(0) = 0.$$

Hence, we obtain

(20)

$$\begin{aligned}\Delta(\rho\circ u)(0) &= \frac{\partial^2(\rho\circ u)}{\partial x^2}(0)+\frac{\partial^2(\rho\circ u)}{\partial y^2}(0)\\ &= \nabla^2\rho(p)\left(\sum_{j=1}^n\frac{\partial u_j}{\partial x}(0),\sum_{j=1}^n\frac{\partial u_j}{\partial x}(0)\right)+\nabla^2\rho(p)\left(\sum_{j=1}^n\frac{\partial u_j}{\partial y}(0),\sum_{j=1}^n\frac{\partial u_j}{\partial y}(0)\right).\end{aligned}$$

□

Finally we have the following

Lemma 5.9. *Let ρ be a C^2 function on M. Assume that for every p in M and every 2-dimensional subspace E of T_pM, there exists a conformal harmonic disc u such that $u(0)=p$ and $T_pu(\mathbb{D})=E$, and such that $\Delta(\rho\circ u)(0)\geq 0$. Then ρ is an MPSH function.*

Proof. Let $f:\mathbb{D}\to M$ be a conformal harmonic immersion. Let $p=f(\zeta_0)$ for some $\zeta_0\in\mathbb{D}$. By assumption, there exists a conformal harmonic disc $u:\mathbb{D}\to M$ satisfying $u(0)=p$, $T_pf(\mathbb{D})=T_pu(\mathbb{D})$, and $\Delta(\rho\circ u)(0)\geq 0$. Let $h=(h_1,h_2)$ be a holomorphic automorphism of $\mathbb{D}$ such that $h(0)=\zeta_0$. Then $g:=f\circ h$ is a conformal harmonic immersed disc, with $g(0)=p$ and $T_pg(\mathbb{D})=T_pf(\mathbb{D})$. Moreover

$$\Delta(\rho\circ g)(0)=\Delta(\rho\circ f)(\zeta_0)\left(\left|\frac{\partial h_1}{\partial x}(0)\right|^2+\left|\frac{\partial h_2}{\partial x}(0)\right|^2\right).$$

Let $v_1:=du(0)\,(\partial/\partial x)$, $v_2:=du(0)\,(\partial/\partial y)$. Let $e_1,\ e_2\in\mathbb{R}^2$, with $\|e_1\|=\|e_2\|=1$ and let $\lambda\in\mathbb{R}$ be such that $dg(0)(e_1)=\lambda v_1$, $dg(0)(e_2)=\lambda v_2$. Let L be a conformal linear isometry of $\mathbb{R}^2$ such that $L(\partial/\partial x)=e_1$, $L(\partial/\partial y)=e_2$. Then $\tilde g:=g\circ L$ is a conformal harmonic immersed disc satisfying $\tilde g(0)=p$, $d\tilde g(0)(\partial/\partial x)=\lambda v_1$, $d\tilde g(0)(\partial/\partial y)=\lambda v_2$ and by Lemma 5.8

$$\Delta(\rho\circ u)(0)=\Delta(\rho\circ\tilde g)(0)=\lambda^2\Delta(\rho\circ g)(0).$$

Hence $\Delta(\rho\circ g)(0)\geq 0$ and so $\Delta(\rho\circ f)(\zeta_0)\geq 0$. □

5.2. **Some examples.** Let $p\in M$ and let ρ be a smooth C^2 function defined in a neighborhood U of p. If ρ is strictly convex on U i.e., ρ is of class C^2 and $\nabla^2\rho$ is positive definite on U, then it follows from (20) that ρ is MPSH on U. In particular, we have

Example 5.10. *Let $x=(x_1,\dots,x_n)$ be normal coordinates at p. Then the function $|x|^2$: $q\mapsto|x(q)|^2=\sum_{j=1}^n x_j^2(q)$ is MPSH in a neighborhood of p.*

Indeed, using (19) we have

$$\nabla^2(|x|^2)=\sum_{i=1}^n\left(1+\mathcal{O}(|x|^2)\right)dx_i\otimes dx_i+\sum_{1\leq i<j\leq n}\mathcal{O}(|x|^2)dx_i\otimes dx_j,$$

which is a small deformation of $\sum_{i=1}^n dx_i\otimes dx_i$ near p. Hence $|x|^2$ is strictly convex near p.

We also have the following

Lemma 5.11. *Suppose that the normal coordinates x are choosen at the point $p=0$. Then the function $\phi(x)=\log|x|+A|x|$ is MPSH in a neighborhood of the origin for a sufficiently large $A>0$.*

Proof. By Lemma 5.5 there exists a family of immersed minimal discs centered at the origin, such that their tangent spaces at the origin fill the Grassmannian $Gr(2,n)$. Furthermore, these discs depend smoothly on the spaces that run over $Gr(2,n)$.

We denote by $\zeta = \xi + i\eta$ the coordinates in $\mathbb{D}$. Let $u : \mathbb{D} \longrightarrow (\mathbb{R}^n, g)$ be a conformal harmonic immersion with $u(\zeta) = (\xi, \eta, 0, ..., 0) + O(|\zeta|^2)$. Then

$$\Delta \log u(\zeta) = O\left(1/|\zeta|\right)$$

and

$$\Delta |u|(\zeta) = (1/|\zeta|)\left(1 + O(|\zeta|)\right).$$

This implies that $\log u + A|u|$ is subharmonic near the origin for a sufficiently large constant $A > 0$. Notice that up to now the constant $A > 0$ depends on u. Hence, there exists $B > 0$ such that

$$\Delta \phi(u(\zeta)) \geq B/|\zeta| \tag{21}$$

near the origin in $\mathbb{D}$. Since everything depends smoothly on parameters, the same estimate holds for minimal discs of the above family close enough to the initial disc u. The tangent spaces of these discs fill an open non-empty subset of $Gr(2,n)$. By compactness of $Gr(2,n)$, we obtain that the estimate (21) holds for all discs with suitable $A, B > 0$. The above family of minimal discs also smoothly depends on it starting point, the origin in our case. Moving (using Lemma 5.5) this family to a point q close enough to the origin, we obtain by continuity that the estimate (21) holds for all minimal discs of this family. This implies Lemma 5.11 in view of Lemma 5.9.

□

6. Localization principle and preliminary boundary estimates

Let (M, g) be a Riemannian manifold of dimension n and let D be a domain in M. We denote by $F_D(x, \xi)$ the value of the Kobayashi-Royden pseudometric at $(x, \xi) \in TM$. The definition of the Kobayashi-Royden metric is similar to the classical case of complex manifolds, see [**?**]. Denote by $\mathbb{B}^n$ the unit ball of $\mathbb{R}^n$. The main result of this section is the following localization principle for the Kobayashi-Royden metric.

Theorem 6.1. *Let $x : U \to 3\mathbb{B}^n$ be a normal coordinate neighborhood in M centered at a point $p \in D$ (in particular, $x(p) = 0$). Assume that U is small enough such that $|x|^2$ is a MPSH function on U. Let ρ be a negative MPSH function in D such that the function $\rho - \varepsilon|x|^2$ is MPSH on $D \cap U$ and $|\rho| \leq B$ in $D \cap x^{-1}(2\mathbb{B}^n)$ for some constants $\varepsilon, B > 0$. Then there exists a constant $C = C(\varepsilon, B) > 0$, independent of ρ, such that for every $w \in D \cap x^{-1}(\mathbb{B}^n)$ and every tangent vector $\xi \in T_w(M)$:*

$$F_D(w, \xi) \geq C|\xi||\rho(w)|^{-1/2}.$$

The coordinate neighborhood U is not assumed to be contained in D. Therefore, this result gives a first (non optimal) asymptotic behavior estimate of F_D near the boundary of D in M. Note also that no conditions such as boundedness or Kobayashi hyperbolicity are imposed on D. A similar rselt is obtained in [**?**] and [**?**] for the case of complex and almost complex manifolds respectively.

The proof consists of several steps.

Step 1. Construction of suitable MPSH functions. Consider a smooth non-decreasing function ψ on $\mathbb{R}_+$ satisfying $\psi(t) = t$ for $0 \leq t \leq 1/2$ and $\psi(t) = 1$ for

$t \geq 3/4$. For each point $q \in M$ satisfying $|x(q)| < 2$, we define the function $\Psi_q = \psi(|x-x(q)|^2)\exp(A\psi(|x-x(q)|))\exp(\lambda\rho)$ on $D \cap U$, and $\Psi_q = \exp(A\psi(|x-x(q)|))\exp(\lambda\rho)$ on $D \setminus U$; the positive constants A and λ will be chosen later.

The function $\log\Psi_q(x) = \log\psi(|x-x(q)|^2) + A\psi(|x-x(q)|) + \lambda\rho$ is MPSH on $D \setminus \{|x-x(q)|^2 \leq 3/4\}$. On the other hand, there exists a constant $A > 0$ such that the function $\log\psi(|x-x(q)|^2) + A|x-x(q)| + A|x|^2$ is MPSH on U. Moreover, by assumption the function $\rho - \varepsilon|x|^2$ is MPSH on $D \cap \{|x-x(q)| \leq 1\}$. Hence, taking $\lambda = A/\varepsilon$, we obtain that the function $\log\Psi_q$ which is MPSH on $D \cap \{|x-x(q)| \leq 1\}$ and, therefore, everywhere on D.

Step 2. Preliminary estimate of the Kobayashi-Royden metric. Let $u : \mathbb{D} \to D$ be a conformal harmonic mapping satisfying $u(0) = q$ with $|x(q)| < 2$. Then the function $\phi(z) = \Psi_q(u(z))/|z|^2$ is subharmonic on the punctured disc $\mathbb{D} \setminus \{0\}$ and is upper bounded by $\exp(A)$ as z tends to the unit circle.

Without loss of generality, we assume that the local coordinates are normal at q. Since u is a conformal map, it is easy to see that $\lim_{z\to 0}\phi(z) = |du(0)e_1)|^2\exp(A\rho(q)/\varepsilon)$. Hence, ϕ extends on $\mathbb{D}$ as a subharmonic function. By the maximum principle for subharmonic functions, we have $|du(0)e_1)|^2 \leq \exp(A)\exp(-A\rho(q)/\varepsilon)$. Now by the definition of the Kobayashi-Royden metric, we obtain the estimate

$$F_D(q,\xi) \geq \exp(-A + A\rho(q)/2\varepsilon)|\xi| \geq N(\varepsilon,B)|\xi| \tag{22}$$

for any $q \in D \cap x^{-1}(2\mathbb{B})$ and $\xi \in T_qD$. Here $N(\varepsilon,B) = \exp(-A-AB/2\varepsilon)$.

Step 3. Localization of the Kobayashi balls on D. As it is shown in [?], F_D is an upper semi-continuous function on the tangent bundle T_D, and d_D is the integrated form of F_D i.e. for any points $p, q \in D$ we have

$$d_D(p,q) = \inf_{\gamma\in\Gamma(p,q)} \int_0^1 F_D(\gamma(t), d\gamma(t))dt \tag{23}$$

where the infimum is taken over the set $\Gamma(p,q)$ of all C^1- smooth paths $\gamma : [0,1] \longrightarrow D$ with $\gamma(0) = p$, $\gamma(1) = q$. Denote by $B_D(q,\delta)$ the Kobayashi ball with respect to d_D, centered at $q \in D$ and of radius $\delta > 0$. We want to compare $B_D(q,\delta)$ with a suitable ball with respect to the Euclidean ball (in local coordinates). This in turn allows to control a distortion of conformal harmonic discs giving the desired localization of the Kobayashi-Royden pseudometric.

Lemma 6.2. *For any point q in $D \cap x^{-1}(\mathbb{B}^n)$ and any $\delta \leq N = N(\varepsilon,B)$, the Kobayashi ball $B_D(q,\delta)$ is contained in $D \cap \{|x-x(q)| < \delta/N\}$.*

Proof. Fix a point $w \in D$. Setting $G = \{\tilde{w} \in U : |x(\tilde{w}) - x(q)| < 1\}$, we obtain from (22) and (23) that

$$d_D(w,q) \geq \inf_{\gamma\in\Gamma(q,w)} \int_{\gamma^{-1}(G)} F_D(\gamma(t), d\gamma(t))dt \geq N \inf_{\gamma\in\Gamma(q,w)} \int_{\gamma^{-1}(G)} |d\gamma(t)|dt.$$

Given a path γ, the last integral represents the Euclidean length of the part of $\gamma([0,1])$ contained in G.

Claim. For $w \in G$, $\inf_{\gamma\in\Gamma(q,w)} \int_{\gamma^{-1}(G)} |d\gamma(t)|dt \geq |x(w) - x(q)|$.

To prove the Claim, there are two cases to consider. First, if the path γ is contained in G, then obviously its length is not smaller than $|x(w) - x(q)|$. Second, if γ intersects the boundary of G, then the length of a connected component of γ joining q and a boundary point of G is larger than or equal to 1, which is larger than or equal to $|x(w) - x(q)|$. This proves the Claim.

Finally, if w is not in G (for example, if w is not in U), then for every $\gamma \in \Gamma(q, w)$, the length $\int_{\gamma^{-1}(G)} |d\gamma(t)|dt$ is bounded from below by 1. Therefore, we have the estimates

$$d_D(w, q) \geq N \min\{1, |x(w) - x(q)|\}, \;\; w \in D \cap U \tag{24}$$

and

$$d_D(w, q) \geq N, \;\; w \in D \setminus U. \tag{25}$$

Now it follows from (24) and (25) that the condition $w \in B_D(q, \delta)$ implies that $w \in U$ and that $|x(w) - x(q)| < \delta/N$. □

Step 4. Precise estimate on F_D. Consider a smooth function ψ defined quite similarly as in Step 1 and satisfying $\psi(t) = t$ for $t \leq 1/2$ and $\psi(t) = 1$ when $t \geq 1$. For every $w \in D \cap x^{-1}(\mathbb{B}^n)$ and every $\lambda, \beta > 0$, consider the function

$$\Phi_{\lambda,\beta,w}(x) = \psi(|x - x(w)|^2/\beta^2) \exp(A\psi(|x - x(w)|)) \exp(\lambda\rho(x))$$

defined on $D \cap U$, where A is given in Step 1. This function is well-defined and takes its values in $[0, e^A]$. There exists a constant $C > 0$ depending only on the function ψ such that the function $\log \Phi_{\lambda,\beta,w} + (C/\beta^2 - \lambda\varepsilon)|x|^2$ is MPSH in $D \cap U$. Now set $\lambda = 1/|\rho(w)|$ and $\beta^2 = C|\rho(w)|/\varepsilon$. We obtain a function denoted by Φ_w, with $\log \Phi_w$ of class MPSH on $D \cap U$.

Set $r = (e^{2N} - 1)/(e^{2N} + 1)$, so that the Poincaré radius of the disc $r\mathbb{D}$ in $\mathbb{D}$ is equal to N. It follows by Lemma 6.2 that for each conformal harmonic mapping $g : \mathbb{D} \longrightarrow D$ such that $w = g(0) \in D \cap x^{-1}(\mathbb{B}^n)$, one has the inclusion $g(r\mathbb{D}) \subset D \cap x^{-1}(2\mathbb{B}^n)$. Let $f : \mathbb{D} \longrightarrow D$ be a conformal harmonic mapping satisfying $f(0) = w$ and $df(0)e_1 = \alpha^{-1}\xi$, where $\alpha > 0$ and $\xi \in T_w D$. Then the function

$$v(z) = \Phi_w(f(z))/|z|^2$$

is subharmonic on $r\mathbb{D} \setminus \{0\}$. Furthermore, $\limsup_{z\to 0} v(z) = \varepsilon|\xi|^2/(eC|\rho(w)|\alpha^2)$ (we choose normal coordinates at w as above). Therefore, v extends on $r\mathbb{D}$ as a subharmonic function. By the maximum principle, we have : $\alpha \geq e^{-A}\varepsilon^{1/2} r|\xi|(eC|\rho(w)|)^{-1/2}$. Now by the definition of the Kobayashi-Royden metric we obtain the estimate

$$F_D((w, \xi) \geq e^{-A}\varepsilon^{1/2} r|\xi|(eC|\rho(w)|)^{-1/2}.$$

This completes the proof of Theorem 6.1.

As a first application, we obtain the following result.

Theorem 6.3. *Let (M, g) be a Riemannian manifold, ρ be a MPSH function on M, and let $u : \mathbb{D} \to M$ be a conformal harmonic map such that $\rho \circ u \geq 0$ on $\mathbb{D}$ and $(\rho \circ u)(z) \to 0$ as $z \in D$ tends to an open arc $\gamma \subset b\mathbb{D}$. Assume that for a certain point $a \in \gamma$ the cluster set $C(f, a)$ contains a point $p \in M$ such that ρ is strictly MPSH near p. Then u extends to a neighborhood of a in $\mathbb{D} \cup \gamma$ as a Hölder 1/2-continuous map.*

The proof follows from Theorem 6.1 via the same argument as in [**?**] so we skip it. In particular, we have the following

Corollary 6.4. *Let ρ be a strictly MPSH function on a Riemannian manifold (M, g). Set $M^+ := \{\rho > 0\}$ and $\Gamma = \{\rho = 0\}$. Assume that $u : \mathbb{D} \to M^+$ is a conformal harmonic disc such that the cluster set $C(u, \gamma)$ is contained in Γ for some open non-empty arc $\gamma \subset b\mathbb{D}$. Then u extends on $\mathbb{D} \cup \gamma$ as a Hölder 1/2-continuous map.*

7. Complete hyperbolicity of strictly pseudoconvex domains

Let (M, g) be a Riemannian manifold. We recall that :

(a) M is (Kobayashi) hyperbolic if the pseudistance d_g is a distance on TM,
(b) M is complete hyperbolic if the metric space (M, d_g) is complete.

Here we obtain one of our main results.

Theorem 7.1. *Let Ω be a relatively compact domain in (M, g). Assume that ρ is a strictly MPSH C^2 function in a neighborhood of $\overline{\Omega}$ such that $\Omega = \{\rho < 0\}$ and $d\rho \neq 0$ on $b\Omega$. Then Ω is a complete hyperbolic domain.*

The proof is based on the approach of S. Ivashkovich - J.P. Rosay in [**?**]. The following lemma is obtained in [**?**] for pseudoholomorphic discs. The proof is the same for conformal harmonic discs, so we drop it.

Lemma 7.2. *Let Ω be a domain in (M, g). Let $p \in b\Omega$. Let ϕ be either:*

(a) *a C^1 map from $\overline{\Omega}$ into $\mathbb{R}^2$ with $\phi(p) = 0$ and $\phi \neq 0$ on Ω, or*
(b) *a C^1 map from a neighborhood U of p int $\mathbb{R}^2$, such that $\phi(p) = 0$ and $\phi \neq 0$ on $U \cap \overline{\Omega} \setminus \{p\}$.*

Let δ be a positive function defined on $(0, +\infty)$, and satisfying $\int_0^1 \frac{dt}{\delta(t)} = +\infty$. Assume that for every conformal hermonic map $u : \mathbb{D} \to \Omega$, such that $u(0)$ is close to p one has

$$|\nabla(\phi \circ u)(0)| \leq \delta(|\phi \circ u)(0)|)$$

(here ∇ denotes the gradient). Then p is at infinite Kobayashi distance from the points in Ω.

We begin with the following localization principle which.

Lemma 7.3. *Let p_0 be a boundary point of Ω. There exists a neighborhood V of p_0 and $r \in (0, 1)$ with the following property: if $u : \mathbb{D} \to \Omega$ is a conformal harmonic map such that $u(0) \in V$, then $u(r\mathbb{D}) \subset V$.*

Consider in the unit disc $\mathbb{D}$ the ball $\mathbb{B}_P(0, t)$ with respect to the Poincare metric, centered at the origin and of radius t. it follows from [**?**, Ga-Su]hat $u(B_P(0, t) \subset \mathbb{B}_\Omega(u(0), t)$. Now result follows by Lemma **??**.

N we present another localization principle for the Kobayashi - Royden metric. In this section ∇ denotes the gradient, and not the Levi-Civita connection.

Lemma 7.4. *Under the hypothesis of Theorem 7.1 let p_0 be a boundary point of Ω. Then for any $0 < r < 1$ there exist $\delta > 0$ and $C > 0$ with the following property: if a conformal harmonic disc $u : \mathbb{D} \longrightarrow \Omega$ satisfies $dist(u(0), p_0) < \delta$, then*

$$dist(u(0), u(\zeta)) \leq C dist(u(0), b\Omega)^{1/2} \tag{26}$$

for $|\zeta| < r$.

Proof. Fix $r < r_1 < 1$ small enough, such that r_1 satisfies the localization Lemma 6.2 . Consider a coordinate neighborhood U of p_0 with normal coordinates centered at $p_0 = 0$. We identify U with a ball in $\mathbb{R}^n$ and assume without loss of generality that the metric g is a small deformation of the standard Euclidean metric g_{st} on that ball. It follows from Example 5.10 and Lemma 5.11 that there exists a neighborhood $V \subset U$ of p_0 and a small enough $\varepsilon > 0$ such that for each $p \in V$ the function $q \mapsto \rho_p(q) = \rho(q) - \varepsilon|q-p|^2$, as well as the functions $q \mapsto |q-p|^2$ are MPSH on V. Here we use the notation $|\cdot|$ for the norm induced on U by the local coordinates x and the Euclidean metric in $\mathbb{R}^n$. Also there exists $A, B > 0$ such that

$$-B|q-p| \leq \rho_p(q) \leq -A|q-p|^2.$$

It follows by the localization principle established in Lemma 6.2 that if $u : \mathbb{D} \to \Omega$ is a conformal harmonic map such that $u(0)$ is close enough to p_0, then $u(\zeta) \in V$ when $|\zeta| \leq r_1$. Choose $p \in b\Omega$ such that $dist(u(0), b\Omega) = dist(u(0), p)$. Since the function $|u(\zeta) - p|^2$ is subharmonic on $|\zeta| \leq r_1$, by the mean inequality we have

$$|u(\zeta) - p|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |u(r_1 e^{i\theta} - p|^2 d\theta.$$

Again by the mean value property we have

$$-(\rho_p \circ u)(0) \geq \frac{1}{2\pi}\int_0^{2\pi} -(\rho_p \circ u)(r_1 e^{i\theta}) d\theta \geq \frac{A}{2\pi}\int_0^{2\pi} |u(r_1 e^{i\theta} - p|^2 d\theta \geq A|u(\zeta) - p|^2.$$

Therefore

$$dist(u(\zeta), u(0)) \leq (dist(u(0), p) + dist(p, u(\zeta)) \leq C dist(u(0), p)^{1/2}$$

for some constant $C > 0$ which proves Lemma 7.4.

We also have the following version of the Schwarz lemma for harmonic map (a more general result is contained in [5], Th. 4.8):

Lemma 7.5. *Let Ω be a bounded subset in $(\mathbb{R}^n, g)$. Let K be a compact subset of Ω. There exists $\delta > 0$ such that: for every $0 < r < 1$ there exists $C > 0$ such that if $u : \mathbb{D} \to \Omega$ is a harmonic disc (with respect to the metric g) and $u(\mathbb{D}) \subset K$, then*

$$|\nabla u(\zeta)| \leq C \sup_{|\omega|<1} |u(\omega) - u(0)| \tag{27}$$

if $|u(\omega) - u(0)| \leq \delta$ and $|\zeta| \leq r$.

We continue the proof of Theorem 7.1. As above, we consider the local normal coordinates centered at $p = 0$. Using dilations, we may assume that U contains the unit ball $\mathbb{B}^n$ and g is close enough to g_{st} in the C^k norm on U, with k big enough. Furthermore, we may assume that $U \cap \overline{\Omega} \setminus \{0\}$ is contained in $\{x \in U : x_1 < 0\}$.

Let $u : \mathbb{D} \to \Omega$ be a conformal harmonic map. We assume that $u(0)$ is close enough to p, so by the localization principle, $u(\zeta) \in U$ when $|\zeta| < r/2$, whee $r \in (0,1)$ is provided by Lemma 6.2. Then by Lemmas 7.4 and 7.5 we obtain

$$|\nabla u(\zeta)| \leq C dist(u(0), b\Omega)^{1/2} \leq C(-u_1(0))^{1/2} \tag{28}$$

if $|\zeta| \leq 1/4$. Rescaling the disc $\mathbb{D}$, we assume that the above estimate holds on $\mathbb{D}$.

Our goal is to prove that changing C if necessary,

$$|\nabla u_1(0)| \leq C|u_1(0)|. \tag{29}$$

Apply the formula (20) to the function $\rho = x_1$. Then for every $\zeta \in \mathbb{D}$

$$\Delta(\rho \circ u)(\zeta) = (\nabla^2 x_1)_{u(\zeta)} \left(\sum_{j=1}^n \frac{\partial u_j}{\partial x}(\zeta), \sum_{j=1}^n \frac{\partial u_j}{\partial x}(\zeta) \right) + (\nabla^2 x_1)_{u(\zeta)} \left(\sum_{j=1}^n \frac{\partial u_j}{\partial y}(\zeta), \sum_{j=1}^n \frac{\partial u_j}{\partial y}(\zeta) \right).$$

Here ∇^2 denote the Hessian defined previously by the Levi-Civita connection.

Using the estimate (28) we conclude that

$$|\Delta u_1(\zeta)| \leq A|u_1(0)| \tag{30}$$

where $A > 0$ is a constant. Consider the function h harmonic on $\mathbb{D}$:

$$h(\zeta) = u_1(\zeta) - \frac{1}{2\pi} \int_{\mathbb{R}^2} \Delta u_1(\omega) \ln(\omega - \zeta) dm(\omega) - A|u_1(0)|. \tag{31}$$

Here we suppose that Δu_1 is extended by 0 outside $\mathbb{D}$. Using (30) we see that

$$\left| \frac{1}{2\pi} \int_{\mathbb{R}^2} \Delta u_1(\omega + \zeta) \ln(\omega) dm(\omega) \right| \leq A|u_1(0)|.$$

Since $u_1 < 0$, we obtain that $h < 0$. Furthermore, we have $|h(0)| < (2A+1)|u_1(0)|$.

Now it follows from the classical Schwarz lemma for negative harmonic functions that $|\nabla h(0)| \leq 2|h(0)|$. Therefore from (31) we obtain

$$|\nabla u_1(0)| \leq |\nabla h(0)| + C \sup |\Delta u_1| \leq (2(2A+1) + CA)\, |u_1(0)|.$$

This proves (29). Now Theorem 7.1 follows exactly as in the proof of Theorem 1 in [**?**], using Lemma 7.2.

As a direct application of Theorem 7.1, we have the following

Corollary 7.6. *Let $\Omega = \{\rho < 0\}$ be a relatively compact domain in a Riemannian manifold (M, g). We assume that ρ is of class C^2 in a neighborhood of $\overline{\Omega}$. If there exists $c > 0$ such that $\nabla^2 \rho(p)(v, v) \geq cg(v, v)$ for every $p \in \Omega$ and every $v \in T_pM$, then Ω is complete hyperbolic.*

As an application of Corollary 7.6 we have the following examples of complete hyperbolic domains in Riemannian manifolds.

Proposition 7.7.
(i) *Let $p \in \mathbb{R}^n$ and let $r > 0$. Then every small C^2 deformation of the Euclidean ball $B_{Eucl}(p, r)$ is complete hyperbolic.*
(ii) *Let (M, g) be a Riemannian manifold. For every $p \in M$, there exists $r > 0$ such that the ball $B_g(p, r)$ is complete hyperbolic.*
(iii) *Let (M, g) be a Riemannian manifold with non positive Riemannian sectional curvature. Then for every $p \in M$ and every $r > 0$, the ball $B_g(p, r)$ is complete hyperbolic.*

Proof. Point (i). For every $p \in \mathbb{R}^n$, the function $f_p : x \mapsto |x-p|^2$ satisfies $\nabla^2 f_p(x)(v,v) \geq \|v\|^2$ for every $x, v \in \mathbb{R}^n$. Let $r' > r$ and let ρ be a C^2 function defined in a neighborhood of $\overline{B_{Eucl}(p,r')}$ and such that $\|\rho - f_p\|_{C^2\left(\overline{B_{Eucl}(p,r')}\right)}$ is sufficiently small. Then there exists $c > 0$ such that for every $x \in B_{Eucl}(p,r')$ and every $v \in \mathbb{R}^n$, $\nabla^2 \rho(x)(v,v) \geq c\|v\|^2$.

Point (ii). It follows from Example 5.10 that there exists a neighborhood U of p and a constant $c > 0$ such that the function $f_p : x \in M \mapsto d_g^2(x,p)$ satisfies, for every $x \in U$ and every $v \in T_xM : \nabla^2 f_p(x)(v,v) \geq cg(v,v)$.

Point (iii). According to Theorem 4.1 (2) in [**?**], the funtion $f_p : x \in M \mapsto d_g^2(x,p)$ is strictly convex in a Riemannian manifold with non positive Riemannian sectional curvature. □

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