Singular Holomorphic Foliations and Levi-flat hypersurfaces

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Portorož, Slovenia, June 2024

1. Holomorphic Foliations

Definition

A holomorphic foliation of codimension 1 on a complex manifold M^n is a decomposition $\mathcal{F}$ of M into pairwise disjoint immersed complex manifolds of dimension $n-1$ (called the *leaves* of $\mathcal{F}$) such that
(i) for any $p \in M$ there exists a unique submanifold L_p of the decomposition that passes through p.
(ii) For any $p \in M$ there exists a *distinguished* holomorphic chart of M (ϕ, U), $p \in U$, $\phi : U \to \phi(U) \subset \mathbb{C}^n$ such that $\phi(U) = \Delta^{n-1} \times \Delta$, and if L is a leaf of $\mathcal{F}$ with $L \cap U \neq \varnothing$, then

$$L \cap U = \cup_{q \in P} \phi^{-1}(\Delta^{n-1} \times \{q\}),$$

where P is a countable subset of Δ.

The sets $\phi^{-1}(\Delta^{n-1} \times \{q\})$ are called plaques of the distinguished chart (ϕ, U). Foliations of arbitrary codimension are defined the same way.

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2. How to define a holomorphic foliation?

A holomorphic foliation can be defined

(0) In prose.
(1) By local submersions.
(2) Foliation of dimension 1 can be defined by a holomorphic vector field: if X is a holomorphic vector field on M, then it defines a foliation on $M \setminus \{X = 0\}$, whose structure is determined by the Flow box theorem for vector fields, so integral curves of X are the leaves. In fact, any foliation of dimension 1 can be locally defined by a holomorphic vector field.
(3) By a holomorphic 1-form $\omega \not\equiv 0$. The form generates a distribution of hyperplanes ker (ω_p) on $M \setminus \{\omega = 0\}$. This defines a foliation tangent to ker (ω_p) iff ω is integrable, i.e., $\omega \wedge d\omega = 0$ (Frobenius). Any foliation of codimension 1 is locally defined by a nonvanishing holomorphic 1-form.

In dimension 2: (2) and (3) are equivalent: $X = P(z,w)\frac{\partial}{\partial z} + Q(z,w)\frac{\partial}{\partial w}$ $\iff \omega = Pdw - Qdz$.

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3. Singular foliations

Let M be a complex manifold, $\dim M \geq 2$.

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A singular holomorphic foliation $\mathcal{F}$ of codimension 1 is given by an open cover $M = \cup_j U_j$, holomorphic integrable 1-forms ω_j on U_j such that if $U_j \cap U_k \neq \varnothing$, then $\omega_j = g_{jk}\omega_k$ in $U_j \cap U_k$ for some $g_{jk} \in \mathcal{O}^*(U_j \cap U_k)$. The singular locus $\text{sing}(\mathcal{F})$ is defined by $\text{sing}(\mathcal{F}) \cap U_j = \{\omega_j = 0\}$. We may assume that $\text{sing}(\mathcal{F})$ is a complex analytic set of codimension ≥ 2.

If $\mathcal{F}$ is a codimension 1 holomorphic foliation on $M \setminus A$, where A is a complex analytic subset of M, codim A ≥ 2, then there exists a singular holomorphic foliation $\tilde{\mathcal{F}}$ on M that extends $\mathcal{F}$.

If $\dim M = 2$ then a singular foliation $\mathcal{F}$ has dimension and codimension 1. In this case $\text{sing}(\mathcal{F})$ is a discrete set and near a singular point $\mathcal{F}$ can be given by a vector field that vanishes precisely at the singularity.

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4. Dimension 2

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}^2$ singular at the origin. A *separatrix* of $\mathcal{F}$ is a holomorphic curve $S \ni 0$ which is invariant by $\mathcal{F}$, i.e., $S \setminus \{0\}$ is contained in a leaf of $\mathcal{F}$. Camacho-Sad (Ann. of Math., 1982): any singularity admits at least one separatrix.

The point 0 is called a *dicritical* singularity if $\mathcal{F}$ has infinitely many distinct separatrices, otherwise the singularity is *nondicritical*.

A nonconstant holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ is called a *holomorphic first integral* for $\mathcal{F}$ if f is constant on the leaves of $\mathcal{F}$.

If $\mathcal{F}$ is given by a vector field X singular at 0, then the above is equivalent to $df(X) \equiv 0$, and in terms of 1-form ω, the condition becomes $\omega \wedge df = 0$.

If $\mathcal{F}$ admits a holomorphic first integral f, then separatrices are the irreducible components of $f^{-1}(f(0))$, so the origin is a nondicritical singularity. The converse is the following.

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5. First integrals

Theorem (Mattei-Moussu, Ann. Sci. Ecole Norm. Sup., 1980)

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}^2$, singular at the origin. If 0 is a nondicritical singularity and the leaves of $\mathcal{F}$ are closed in $V \setminus \{0\}$ for some neighbourhood V of the origin, then there exists a neighbourhood $W \subset V$ of 0 and a holomorphic first integral $f : W \to \mathbb{C}$ of $\mathcal{F}$.

What about dicritical singularities? These clearly cannot admit holomorphic first integrals, but may have meromorphic first integrals, i.e., a nonconstant meromorphic function g with the indeterminacy at the origin that is constant on the leaves of F.

Example: The foliation corresponding to $\omega = zdw - wdz$ has leaves that are complex lines passing through the origin, which is a dicritical singularity. The meromorphic first integral is $g(z, w) = z/w$.
There are examples of dicritical singularities that do not admit meromorphic first integral, Cerveau-Mattei (Asterisque, 1982):topologically indistinguishable).

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6. Levi-flat hypersurfaces 1

Let $M \subset \mathbb{C}^n$ be a C^∞-smooth real hypersurface defined by $\rho = 0$ with $d\rho|_M \neq 0$. Then the following are equivalent:

(i) M is foliated by complex hypersurfaces.
(ii) The complex hyperplane distribution $H_p \subset T_pM$ is involutive on M.
(iii) The Levi-form $\frac{\partial^2 \rho(p)}{\partial z_j \partial \bar{z}_k}|_{H_p} = 0$ for all $p \in M$.

The foliation of M by complex hypersurfaces is called the *Levi foliation* of M.

Cartan: if M is real analytic Levi flat, then M is locally biholomorphically equivalent to $\{x_n = 0\}$. Other examples: For a curve $\gamma \subset \mathbb{C}$, $\gamma \times \mathbb{C}^k$ is Levi flat; if $f : \mathbb{C}^n \to \mathbb{C}_\xi$ is a submersion, then $f^{-1}(\mathrm{Re}\,\xi)$ is Levi-flat.

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Let M be an irreducible real analytic set in $\mathbb{C}^n$ of codimension 1. M is called Levi-flat, if the regular locus of M (of dimension $2n-1$) is Levi-flat (in the above sense).

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Let M be an irreducible real analytic set in $\mathbb{C}^n$ of codimension 1. M is called Levi-flat, if the regular locus of M (of dimension $2n - 1$) is Levi-flat (in the above sense).

6. Levi-flat hypersurfaces 1

Let $M \subset \mathbb{C}^n$ be a C^∞-smooth real hypersurface defined by $\rho = 0$ with $d\rho|_M \neq 0$. Then the following are equivalent:

(i) M is foliated by complex hypersurfaces.

(ii) The complex hyperplane distribution $H_p \subset T_pM$ is involutive on M.

(iii) The Levi-form $\frac{\partial^2 \rho(p)}{\partial z_j \partial \overline{z}_k}|_{H_p} = 0$ for all $p \in M$.

The foliation of M by complex hypersurfaces is called the *Levi foliation* of M.

Cartan: if M is real analytic Levi flat, then M is locally biholomorphically equivalent to $\{x_n = 0\}$. Other examples: For a curve $\gamma \subset \mathbb{C}$, $\gamma \times \mathbb{C}^k$ is Levi flat; if $f : \mathbb{C}^n \to \mathbb{C}_\xi$ is a submersion, then $f^{-1}(\text{Re } \xi)$ is Levi-flat.

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7. Levi-flat hypersurfaces 2

Example 1: $M_1 = \{z \in \mathbb{C}^n : \text{Re } (z_1^2 + \cdots + z_n^2) = 0\}$ is a real algebraic Levi flat hypersurface singular at the origin. Burns-Gong (Amer. J. of Math., 1999): if a real analytic hypersurface

$$M = \{z \in \mathbb{C}^n : \text{Re } (z_1^2 + \cdots + z_n^2) + O(|z|^3) = 0\}$$

is Levi flat, then M is biholomorphically equivalent to M_1 (cf. Cartan.)

Example 2: the hypersurface $M_2 = \{z \in \mathbb{C}^2 : |z_1|^2 - |z_2|^2 = 0\}$ is Levi flat, singular at 0; Again, Burns-Gong proved that if a real analytic Levi flat M is given by $\rho = Q(z, \overline{z}) + O(|z|^3)$, where Q is positive definite with the rank of the Levi form at least 2, then M is biholomorphic to M_2. Note that the Levi foliation of M_2 agrees with the previously discussed singular foliation on $\mathbb{C}^2$ given by the 1-form $\omega = zdw - wdz$.

The above results of Burns-Gong give a classification of Levi flat hypersurfaces with an isolated nondegenerate singularities. Further generalizations can be found in the work of Fernández-Pérez and co-authors. But a general local classification of singular Levi flat hypersurfaces in incomplete.

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8. More on first integrals

Definition

A (singular) Levi flat hypersurface M is called tangent to a (singular) holomorphic foliation $\mathcal{F}$ if the leaves of the Levi foliation on M are also the leaves of $\mathcal{F}$.

Theorem (Cerveau-Lins Neto, Amer. J. of Math., 2011)

Let $\mathcal{F}$ be the germ at $0 \in \mathbb{C}^n$, $n \geq 2$, of a holomorphic codimension one foliation which is tangent to the germ at $0 \in \mathbb{C}^n$ of a real analytic Levi flat M. Then $\mathcal{F}$ has a nonconstant meromorphic first integral. If $n = 2$, then further
(a) If $\mathcal{F}$ is dicritical then it has a meromorphic first integral;
(b) If $\mathcal{F}$ is nondicritical then it has a holomorphic first integral.

The problem can be reduced to the case $n = 2$. In this dimension Brunella (L'Enseignement Math. 2012) gave an elegant geometric proof.

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We will see later that the result of Cerveau - Lins Neto is a proper generalization of Mattei-Moussu.

Perhaps unnoticed in the literature, the statement of the theorem above is, in fact, "if and only if". The following argument is essentially due to Burns-Gong:

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}^n$ singular at the origin. Suppose it admits a meromorphic first integral m. Near the origin we set $m = f/g$, where $f(0) = g(0) = 0$, and f and g are holomorphic and coprime. Then

$$M = \{z \in \mathbb{C}^n : \text{Re}\ (f(z)\overline{g}(z)) = 0\}$$

is a real analytic Levi flat set. One readily sees that M is tangent to $\mathcal{F}$.

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9. Even more on first integrals

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10. Segre varieties 1

Let $M \subset \mathbb{R}^n_x$ be the germ of a real analytic set at the origin. The complexification M^c of M is a complex analytic germ at the origin in $\mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$ such that that any holomorphic function that vanishes on M necessarily vanishes on M^c. Equivalently, M^c is the smallest complex analytic germ in $\mathbb{C}^n$ that contains M. If $\dim M = 2n - 1$, then it admits a defining function ρ such that $M^c = \{\rho^c = 0\}$, where ρ^c is a complexification of ρ, and ρ^c is *minimal*, i.e., is a generator of the sheaf of ideals on M^c. Complexification of real analytic sets in $\mathbb{C}^n$ can be obtained by a totally real embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^{2n}$.
Given a real analytic hypersurface M and using a minimal defining function of M as above, in some small neighbourhood U of the origin, for $w \in U$ the Segre variety Q_w (associated with M) is defined by

$$Q_w = \{z \in U : \rho(z, \overline{w}) = 0\}.$$

Q_w is a complex hypersurface in U for most $w \in U$.

11. Segre varieties 2

Definition

A point $w \in M \subset \mathbb{C}^n$ is called *Segre degenerate* if $\dim Q_w = n$.

For example, if $M = \{|z_1|^2 - |z_2|^2 = 0\}$, then $Q_w = \{z_1\overline{w}_1 - z_2\overline{w}_2 = 0\}$, and so $w = 0$ is a Segre degenerate point.

Let now M be a (singular) Levi flat hypersuface. Denote by L_p the leaf of the Levi foliation that passes through a regular point $p \in M$. Then there exists a unique irreducible component S_p of Q_p containing L_p. This is also a unique complex hypersurface through p which is contained in M.

A singular point $0 \in M$ is called *dicritical* if it belongs to infinitely many geometrically different leaves L_p.

Theorem (Pinchuk-Sukhov-RS, Izv. Math., 2017)

A singular point of a Levi flat M is dicritical iff it is Segre degenerate

If 0 is Segre degenerate, then $0 \in Q_w$ for all w, but one needs to show that $0 \in S_w$. Our proof essentially shows that, in fact, every irreducible component of Q_w passes through the origin. This shows that 0 is dicritical.

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12. Levi flat hypersurfaces as minimal sets

Given a (singular) holomorphic foliation $\mathcal{F}$ on a compact complex manifold M, a closed set S is called *nontrivial minimal* if it is 1) invariant for $\mathcal{F}$ (or saturated in $\mathcal{F}$), i.e., if $p \in S$, then $L_p \subset S$; 2) $S \neq \varnothing$; 3) minimal wrt inclusion; 4) (nontriviality) does not contain singular points of $\mathcal{F}$.

Theorem (Lins Neto (Ann. de L'Inst. Fourier, 1999)

No codimension 1 holomorphic foliation on $\mathbb{CP}^n$, $n > 2$, admits a nontrivial minimal set.

This implies nonexistence of (closed nonsingular) Levi flat hypersurfaces in $\mathbb{CP}^n$ for $n > 2$, because such a Levi flat can be realized as a nontrivial minimal set of some holomorphic foliation on $\mathbb{CP}^n$. This can be also proved directly as follows.

Let M be a nonsingular closed real analytic Levi flat in $\mathbb{CP}^n$. Then there exists a neighbourhood U of M in $\mathbb{CP}^n$ such that the Levi foliation on M extends to a nonsingular holomorphic foliation on U.

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13. Nonexistence of Levi flats

Lins Neto: if V is a Stein manifold and $K \subset V$ is a compact satisfying $U = V \setminus K$ is connected, Then any holomorphic codimension one foliation $\mathcal{F}$ on U, such that codim $(sing(\mathcal{F})) \geq 2$, can be extended to a holomorphic foliation on V.

The complement of M is the union of two Stein manifolds, and so the Levi foliation on M extends to a (singular) holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}^n$, and M is an invariant subset of $\mathcal{F}$. Lins Neto: Sing($\mathcal{F}$) necessarily contains a component of codimension 2.

A Stein manifold has no complex varieties of positive dimension. Therefore, if $n > 2$, Sing$(\mathcal{F}) \cap M \neq \varnothing$, which is a contradiction. Hence, there are no nonsingular Levi flats in $\mathbb{CP}^n$ for $n > 2$. If $n = 2$, the singularities of $\mathcal{F}$ are isolated points and the argument falls short.

Further generalizations of nonexistence of Levi flats were obtained by Brunella, Ohsawa, Brinkschulte, and others.

The problem is open for $\mathbb{CP}^2$.

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14. Extension of Levi foliations

As we already saw, a Levi foliation on M extends to a holomorphic foliation in a neighbourhood of any regular point on M. A natural question is whether this holds near singular points of a Levi flat M.

The answer is no in general.

Example (Brunella, Ann. Sc. Norm. Super. Pisa, 2007): Let

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : y_2^2 = 4(y_1^2 + x_2)y_1^2\}, \quad z_j = x_j + iy_j.$$

This is a singular Levi flat hypersurface with the Levi foliation $\mathcal{L}$ given by

$$L_c = \{z_2 = (z_1 + c)^2, y_1 \neq 0\}, \quad c \in \mathbb{R}.$$

It cannot be extended to a neighbourhood of the origin as a (singular) holomorphic foliation due to ramification. The extension of $\mathcal{L}$ is a "foliation with branching" given by solutions of ODE $(\frac{dz_2}{dz_1})^2 = 4z_2$.

(i.e., the leaves are graphs of functions $z_2 = z_2(z_1)$ satisfying the ODE.)

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As we already saw, a Levi foliation on M extends to a holomorphic foliation in a neighbourhood of any regular point on M. A natural question is whether this holds near singular points of a Levi flat M. The answer is no in general.

Example (Brunella, Ann. Sc. Norm. Super. Pisa, 2007): Let

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : y_2^2 = 4(y_1^2 + x_2)y_1^2\}, \quad z_j = x_j + iy_j.$$

This is a singular Levi flat hypersurface with the Levi foliation $\mathcal{L}$ given by

$$L_c = \{z_2 = (z_1 + c)^2, y_1 \neq 0\}, \quad c \in \mathbb{R}.$$

It cannot be extended to a neighbourhood of the origin as a (singular) holomorphic foliation due to ramification. The extension of $\mathcal{L}$ is a "foliation with branching" given by solutions of ODE $(\frac{dz_2}{dz_1})^2 = 4z_2$.

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15. Holomorphic d-webs

For simplicity, consider only the case $n = 2$. For $d \in \mathbb{N}$, a singular holomorphic d-web $\mathcal{W}$ is defined by solutions of ODE

$$P\left(z_1, z_2, \frac{dz_2}{dz_1}\right) = \sum_{k=0}^{d} a_k(z_1, z_2) \left(\frac{dz_2}{dz_1}\right)^k = 0,$$

where $a_k(z_1, z_2)$ are holomorphic in a neighbourhood of 0. We treat z_2 as a function of z_1. If the above expression admits factorization into holomorphic terms linear in dz_2/dz_1, then the web splits into the union of d foliations. In particular, if $d = 1$ we obtain a usual foliation. But in general, the ODE is unresolved with respect to dz_2/dz_1.

So at a regular point p of $\mathcal{W}$ there are exactly d leaves of the web that pass through p.

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16. Extension of Levi foliations as webs

A holomorphic correspondence $G : \mathbb{C}^n \to \mathbb{C}^m$ is a multiple-valued map whose graph is a complex analytic set in $\mathbb{C}^n \times \mathbb{C}^m$ with a proper projection on $\mathbb{C}^n$. A meromorphic correspondence is defined the same way except that we allow the fibres of the projection $\Gamma_G \mapsto \mathbb{C}^n$ from the graph of G to be of positive dimension on a set of codimension 2 in $\mathbb{C}^n$.

Example: $G(z_1, z_2) = \sqrt{z_1}$, $G(z_1, z_2) = \sqrt{z_1}/z_2$.

We say that $G : \mathbb{C}^2 \to \mathbb{C}$ is a multiple-valued holomorphic (meromorphic) first integral for a d-web $\mathcal{W}$ if for every regular point p of $\mathcal{W}$ for any leaf L_p there exists a branch of G near p that is constant on L_p.

Theorem (Sukhov-RS, Comment. Math. Helv., 2015)

Let $M \subset \mathbb{C}^n$ be a real analytic Levi-flat, $0 \in M$ is a singular point such that 0 is nondicritical, or M is real algebraic. Then there exist a neighbourhood U of the origin and a singular holomorphic d-web $\mathcal{W}$ in U such that $\mathcal{W}$ extends the Levi foliation of M. Furthermore, $\mathcal{W}$ admits a multiple-valued meromorphic first integral in U .

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17. First integrals and Segre varieties

As we see from the discussion above, the existence of a first integral for a given foliation in intricately related to the existence of a tangent Levi flat hypersurface.

For a further speculation, consider the so-called Segre map: $\lambda : w \to Q_w$ associated with a Levi flat hypersurface M. This is a formal map from a neighbourhood U of a (singular) point in M into the (formal) space of Segre varieties defined in U. If $p \in M \cap U$ is a regular point, and L_p is the leaf of the Levi foliation on M passing through p, then $L_p \subset Q_p$, and one can show that, in fact, $Q_p = Q_w$ for any point $w \in L_p$.

It follows then that the map λ is constant on the leaves of the foliation. Therefore, if one can realize λ as a holomorphic map from U into $\mathbb{C}$, then λ is a holomorphic first integral!

The family of Serge varieties associated with a Levi flat hypersurface is complex one dimensional, and so this idea can be realized. This is the main idea behind Brunella's proof of the Cerveau-Lins Neto theorem, and the proof of Sukhov-RS for d-webs.

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18. More on Mattei-Moussu

Proposition

Let $\mathcal{F}$ be a holomorphic foliation on $\mathbb{C}^2$, singular at 0. The following are equivalent.
(a) 0 is a nondicritical singularity and the leaves of $\mathcal{F}$ are closed in $V \setminus \{0\}$ for some neighbourhood V of the origin.
(b) There exists a neighbourhood W of 0 and a holomorphic first integral $f : W \to \mathbb{C}$ of $\mathcal{F}$.
(c) There exists a (singular) Levi flat hypersurface M tangent to $\mathcal{F}$.

Mattei-Moussu proved $(a) \Rightarrow (b)$. The implication $(a) \Leftarrow (b)$ is immediate. That (b) implies (c) can be shown by taking the preimage of some real line under the first integral. Finally, $(c) \Rightarrow (a)$ follows from the observation that a leaf of the foliation $\mathcal{F}$ is a component of a Segre variety of M and the properties of Segre varieties.

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19. Local hulls of Levi flat hypersurfaces, 1

Smooth case: A smooth real hypersurface M is locally polynomially convex iff M is Levi-flat.

Indeed, if the Levi form of M has at least one nonzero eigenvalue, then there exist holomorphic discs attached to M. If M is Levi flat real analytic, then the proof of local polynomial convexity is straight forward; the general case (when M is just C^1-smooth) can be found in Airapetyan (Math USSR Sbornik,1989).

Note that a Levi flat may contain boundaries of "large" holomorphic discs, e.g., $\mathbb{D} \times \{pt\}$ are attached to $S^1 \times \mathbb{C}$.

What about local convexity near singular points of a real analytic Levi flat?

Suppose that $0 \in M$ is a Segre nondegenerate (nondicritical) singular point. Then by the previous result, the Levi foliation extends as a d-web. We say that the Segre singularity is *unbranched* if $d = 1$, i.e., the Levi foliation extends as a singular foliation to the ambient space.

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20. Local hulls of Levi flats, 2

Theorem (Sukhov-RS, Int. J. of Math., 2021)

Let $M \subset \mathbb{C}^n$ be a real analytic Levi-flat hypersurface, $0 \in M$ (*). Then

(i) *If 0 is a regular point of M, or is an unbranched Segre nondegenerate singularity, then M is locally polynomially convex at 0.*

(ii) *There exists $M \subset \mathbb{C}^2$ with a two-branched Segre nondegenerate singularity at 0 that is not locally rationally convex at 0. Furthermore, there exists a neighbourhood basis $(U_k)_{k \in \mathbb{N}}$ of 0 such that for every k the polynomially convex hull of $M \cap U_k$ contains a full neighbourhood of 0 in $\mathbb{C}^2$.*

(iii) *If 0 is a Segre degenerate (dicritical) singularity, then M is not locally rationally convex at 0. Furthermore, the rationally convex hull of any compact neighbourhood $K \subset M$ of 0 includes a family of Riemann surfaces with the following property: each surface is not contained in M, but its boundary is contained in K.*

21. Hulls of Levi flats 3

(*) In this result we ignore possible points on M where it is a real analytic manifold of dimension $< 2n - 1$, i.e., we consider the set $\overline{M^{\text{reg}} \setminus M^{\text{sng}}}$, even if this is only a semianalytic set. In particular, $0 \in \overline{M^{\text{reg}}}$.

Part (i): in the situation of the theorem, the singular foliation admits a holomorphic first integral $f : U \to \mathbb{C}$ in some neighbourhood of the origin in $\mathbb{C}^2$. The set $f(M)$ is subanalytic (in fact, semianalytic), and therefore, the complement of $f(M)$ in some neighbourhood of $f(0) \in \mathbb{C}$ is connected. To prove local polynomial convexity of $0 \in M$ one can use Oka's characterization of polynomial convexity using the level sets of f that avoid the set $f(M)$.

Part (iii): one can show that there exists w_0 near 0 such that $Q_{w_0} \cap M = \{0\}$. Then a small perturbation S of Q_{w_0} will be such that $M \cap S$ bounds a domain in S, which is a complex analytic variety attached to M. This variety is part of the polynomially (rationally) convex hull of a compact neighbourhood of 0 that contains $M \cap S$.

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22. Hulls of Levi flats 4

Part (ii): the example of a Levi flat M with a nonempty local hull is the example of Brunella discussed above. One can show that the map $f(z_1, z_2) = z_1 \pm \sqrt{z_2}$ is the multiple valued first integral. One can prove that the closure of smooth points can be given by $\{\mathrm{Im}\ (z_1 \pm \sqrt{z_2}) = 0\}$. The map $F : \mathbb{C}^2_w \to \mathbb{C}^2_z$ given by $(w_1, w_2) \to (w_1, w_2^2)$ satisfies

$$F^{-1}(M) = \{\mathrm{Im}\ (w_1 + w_2) = 0\} \cup \{\mathrm{Im}\ (w_1 - w_2) = 0\}.$$

After a complex linear change of coordinates we may assume that the hyperplanes above have the form $\{y_j = 0\}$, $j = 1, 2$, with the intersection equal to $\mathbb{R}^2$. The domain $\mathbb{C}^2 \setminus \mathbb{R}^2$ is given by the union of 4 domains $\{\pm y_j < 0\}$. As an example, the wedge $W = \{y_j < 0, j = 1, 2\}$ is contained in the strictly pseudoconvex domain

$$\Omega = \{y_1 + y_2 + y_1^2 + y_2^2 < 0\},$$

which is biholomorphic to the unit ball.

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Then there exists a complex curve touching the boundary Ω from outside exactly at the origin. Translating this curve and considering the intersections with W gives a family of Riemann surfaces filling W. This proves the result.

Corollary

Let $M \subset \mathbb{C}^n$, $n > 1$, be a Levi-flat hypersurface such that its Levi foliation extends as a singular foliation to a neighbourhood of a singular point $p \in M$. Then M is locally polynomially convex at p if and only if p is a Segre nondegenerate (nondicritical) singularity of M.

We do not know if every Levi flat hypersurface that admits the extension of the Levi foliation as a d-web with $d > 1$ has a nontrivial hull.

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A special case of a d-web in $\mathbb{C}^2$ is when the defining pseudopolynomial is monic, i.e., $\mathcal{W}$ is defined by

$$\left(\frac{dz_2}{dz_1}\right)^d + a_{d-1}(z_1, z_2)\left(\frac{dz_2}{dz_1}\right)^{d-1} + \cdots + a_0(z_1, z_2) = 0. \qquad (1)$$

If $\mathcal{W}$ is given by (1) then
(i) $\mathcal{W}$ admits a multiple-valued holomorphic first integral.
(ii) There exists a singular Levi flat tangent to $\mathcal{W}$.
(iii) There exists an irreducible complex analytic set $A \subset \mathbb{C}^2 \times \mathbb{C}$ of dimension 2 such that the regular part of A is foliated by complex curves and the projection $\pi : A \to \mathbb{C}^2$ sends this foliation to $\mathcal{W}$.

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Thank you! Hvala vam!