The Ru-Vojta inequality and its applications

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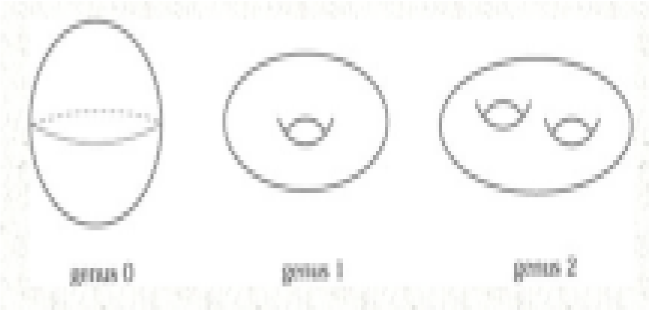
1. compact Riemann surfaces of genus ≥ 2.
2. A torus $T \setminus \{\text{a point}\}$
3. $\mathbb{P}^1 \setminus \{3 \text{ points}\}$



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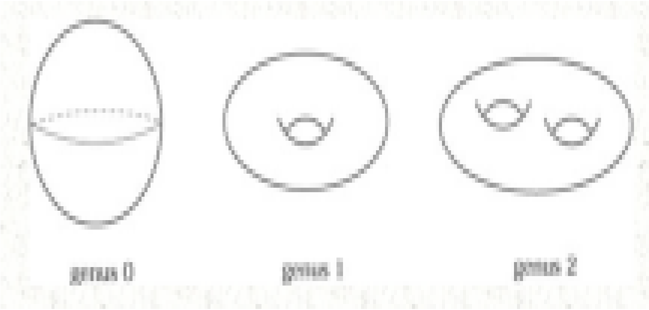

K_M=canonical line bundle. Riemann Roth Theorem
$\deg K_M = 2(g(M)-1)$, so K_M^{-1} ample $\iff g(M)=0$, $K_M = \mathcal{O}_M$ $\iff g(M)=1$, K_M ample $\iff g(M) \geq 2$.

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$$T_{f,D}(r) = m_f(r, D) + N_f(r, D) + O(1).$$

where $T_{f,D}(r) := \int_1^r \frac{dt}{t} \int_{|z|<t} f^* c_1([D])$, $N_f(r, D) := \int_1^r n_{f,D}(t) \frac{dt}{t}$, and $m_f(r, D) = -\int_0^{2\pi} \log \|s_D(f(re^{i\theta})\| \frac{d\theta}{2\pi}$.

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$$(q-(n+1))T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{[n]}(r, H_j)$$
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- **Siu-Yeung, 1997, Noguchi, Winkelmann and Yamanoi, 2002**: Let A be an abelian variety and D be an ample divisor on A. Let $f : \mathbb{C} \to A$ be holomorphic with $f(\mathbb{C}) \not\subset D$. Then $T_{f,D}(r) \leq_{exc} N_f^{(1)}(r, D) + C(\log^+ T_{f,D}(r) + \delta \log r) + O(1)$.

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In the case when $D_j \sim A$, then $\beta(D, D_j) = \frac{q}{n+1}$, where $D = D_1 + \cdots + D_q$.

Theorem (Ru-Vojta, 2020, Arithmetic Part) Let X be a projective variety over a number field k, and $D_1, \dots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle on X with $h^0(L^N) \geq 1$ for N big enough. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality

$$\sum_{i=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq (1+\epsilon) h_L(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

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Note: The GCD problem eventually gets to to estimate $N_f(Y, r)$ (or $T_{f,Y}(r)$ or $h_Y(x)$ in the arithmetic case) for closed subscheme Y with $\mathrm{codim} Y \geq 2$.

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The use of the GCD theorem (producing F and G)

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Theorem[Guo-Sun-W.] Let $g_0, g_1, \ldots, g_n$ be nonconstant units and $\mathbf{g} = (g_0, \ldots, g_n) : \mathbb{C} \to \mathbb{P}^n$. Let G be a nonconstant homogeneous polynomials in $K_{\mathbf{g}}[x_0, \ldots, x_n]$ with no repeated nonmonomial factors in $K_{\mathbf{g}}[x_0, \ldots, x_n]$. Let $\epsilon > 0$. If $g_0, \ldots, g_n$ are multiplicatively independent modulo $K_{\mathbf{g}}$, then
$N_{G(\mathbf{g})}(0, r) - N^{(1)}_{G(\mathbf{g})}(0, r) \le_{\mathrm{exc}} \epsilon T_{\mathbf{g}}(r)$.

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Additionally, if $n = 2$, then $\sum_{i=1}^{3} \delta_{\mathbf{f}}^{(1)}(D_i) < 3$, where, for a divisor D with $d = \deg(D)$, $\delta_{\mathbf{f}}^{(1)}(D) = 1 - \limsup_{r \to \infty} \frac{N_f^{(1)}(r, D)}{d T_f(r)}$.

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This can be seen as a generalization of Borel's Theorem stating that nowhere vanishing entire functions $h_1, \ldots, h_{n+1}$ satisfying the identity $h_1 + \cdots + h_{n+1} = 1$ are dependent.
• Corollary. Let $h_1, \ldots, h_n$ be holomorphic functions on $\mathbb{C}$ such that $\frac{1}{(h_1 \cdots h_n) \cdot (1 - \sum_{j=1}^n h_i)}$ is holomorphic. Then $h_1, \ldots, h_n$ are algebraically dependent.