On the dynamics of complex Hénon maps

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based on joint work with Tanya Firsova and Raluca Tanase

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Polynomial automorphism of $\mathbb{C}^2$

Theorem (Friedland, Milnor 1989)

Every polynomial automorphism of $\mathbb{C}^2$ is conjugate by a polynomial automorphism to one of the following maps:

(a) *affine maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ k' \end{pmatrix}$, $ad - bc \neq 0$

(b) *elementary maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} ax + p(y) \\ by + c \end{pmatrix}$, $ab \neq 0$

(c) *compositions of Hénon maps* $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} p(x) - ay \\ x \end{pmatrix}$, $a \neq 0$

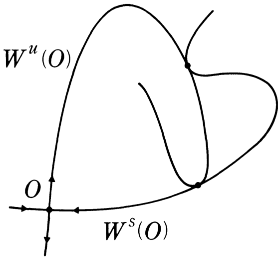
Universality of the Hénon family

Berger, Palis, Takens: Hénon-like maps

$$f_{c,a}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x^2+c-ay\\x\end{pmatrix}+g(x,y).$$

where g has small norm, appear in unfoldings of homoclinic tangencies between the stable and unstable manifold of a saddle periodic point in dissipative systems with one unstable Lyapunov exponent.

The study of part of the local dynamics in these unfoldings is reduced to the study of the dynamics of Hénon-like maps.



Hénon maps

A complex Hénon map $H_{c,a} : \mathbb{C}^2 \to \mathbb{C}^2$ is a diffeomorphism of $\mathbb{C}^2$

$$H_{c,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c - ay \\ x \end{pmatrix}, \quad a \neq 0.$$

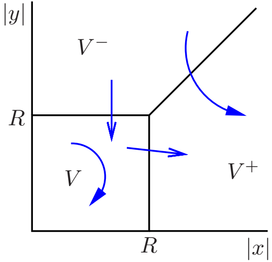
Dynamical objects:

$$K^{\pm} = \text{points in } \mathbb{C}^2 \text{ with bounded forward/backward orbit}$$

$$J^{\pm} = \partial K^{\pm} \;\text{ and }\; J = J^{-} \cap J^{+}$$

The sets J and $J^{\pm}$ are the Julia sets of the Hénon map.

Hubbard's dynamical filtration of $\mathbb{C}^2$



forward escaping set

$$U^+ = \bigcup_{k\geq 0} H^{-\circ k}(V^+)$$

backward escaping set

$$U^- = \bigcup_{k\geq 0} H^{\circ k}(V^-)$$

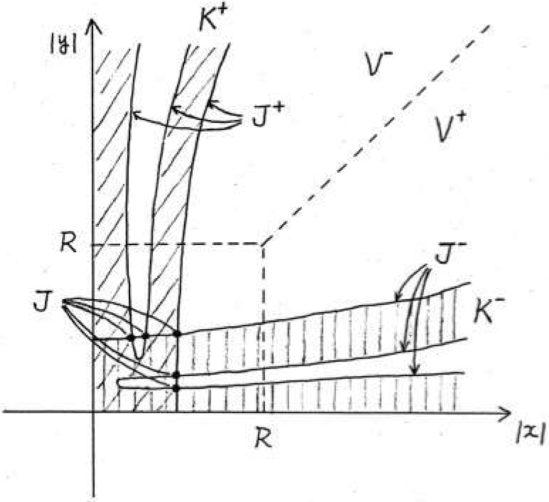


Fig.: An interpretation of the dynamical filtration by Yutaka Ishii.

Some milestones (general, for all Hénon maps)

- In 1982 Calabi proposed to investigate the basin of attraction of the Hénon map as a candidate for a Fatou-Bieberbach domain.
- Hubbard set the ground for the study of the complex Hénon map in 1986. Hubbard-Oberste-Vorth 1990-93 classify the analytic structure of $U^+ = (\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C} / \Gamma_{p,a}$, a quotient by a discrete subgroup of Aut$((\mathbb{C} - \overline{\mathbb{D}}) \times \mathbb{C})$ isomorphic to $\mathbb{Z}[\frac{1}{2}]/\mathbb{Z}$.
- Fornæss-Sibony 1989-1992 developed pluripotential theory for Hénon maps.
- Bedford-Smillie 1991-2002 have many papers on the subject. Among their many results is the general criterion for the connectivity of the Julia set: J is connected if and only if $W^u(p) \cap K^+$ is connected for any saddle periodic point p.
- Lyubich-Radu-Tanase 2016 studied the case of Hénon maps which are not linearizable (analytically) near a semi-neutral fixed point. They proved the existence of Siegel compacta (hedgehogs) inside the center manifolds of the fixed point; the dynamics on the Siegel compacta is recurrent. [local dynamics]

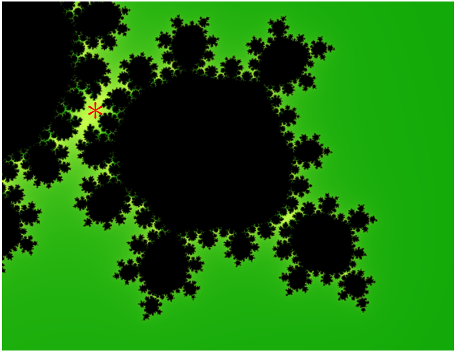


Fig.: Bedford-Smillie's unstable connectivity. The black region represents K^+ restricted to the unstable manifold. We notice an unstable critical point (in red) of the Green function $G^+|_{W^u(\mathbf{q})}$ and that $W^u(\mathbf{q}) \cap K^+$ is disconnected.

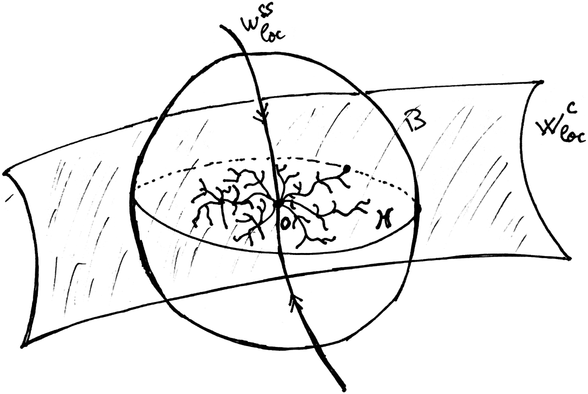
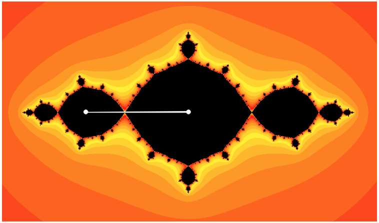


Fig.: Sketch of a hedgehog $\mathcal{H}$ inside a center manifold. In fact, all center manifolds of the origin contain the hedgehog.

Basic dynamical properties of Hénon maps

- K^+ is unbounded, connected, closed, not compact, invariant; K is compact.
- J^+, J^- are always connected, but J may or may not be.
- Friedland-Milnor: in the dissipative case K^- has no interior, so $K^- = J^-$.
- The Hénon map has finitely many periodic points of period n. In the dissipative case, all attracting periodic points are in the interior of K^+ and their basins of attraction are Fatou-Bieberbach domains. The common boundary of each basin is J^+ (which is nowhere a topological manifold except in some very specific cases).

A concrete example and a topologial model for a Fatou-Bieberbach domain. Consider complex Hénon maps which are singular perturbations of $z \mapsto z^2 - 1$.



Theorem (R.)

There are two connected components of the interior of K^+, each a Fatou-Bieberbach domain in $\mathbb{C}^2$. Their boundary J^+ is homeomorphic to $\mathbb{S}^3 - \Sigma$ factored by an equivalence relation, where Σ is a dyadic solenoid.

More milestones (specific dissipative families)

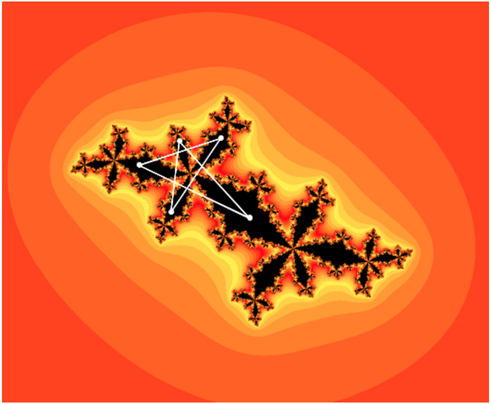
- Small perturbations of polynomials: a lot of progress has been made by Benedicks-Carleson (existence of strange attractors), Berger (perturbations of a composition of two quadratic polynomials), Avila (existence of two coexisting attractors), Hubbard-Oberste-Vorth (hyperbolic perturbations with connected Julia), Fornæss-Sibony (hyperbolic perturbations with disconnected Julia), Lyubich-Martens-de Carvalho and Lyubich-Crovisier-Yang-Pujals (renormalization, existence of Cantor attractor, study of infinitely renormalizable maps), Radu-Tanase (structure of perturbations of parabolic polynomials and continuity of Julia sets), Yampolsky-Yang, Gaidashev-Radu-Yampolsky (perturbations of Siegel polynomials), Bedford-Smillie-Ueda (parabolic implosion, discontinuty of Julia sets), etc.
- mild restrictions on the Jacobian ($|a| < 1/4$): Lyubich-Peters, Lyubich-Dujardin, Bedford-Dujardin on the structure of the interior of K^+. They assume global partial hyperbolicity.

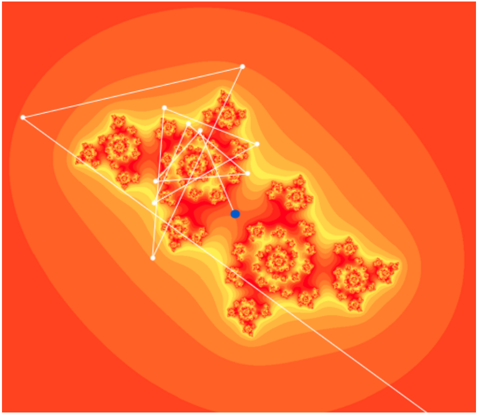
Do critical points play any role?

1D: Critical points play a fundamental role for the dynamics of polynomials.

- The Julia set of a polynomial is connected if and only if all critical points have bounded forward orbits.
- A polynomial is hyperbolic if and only if the closure of the orbits of the critical points are disjoint from the Julia set.
- Easy to plot in the parameter space.
- A polynomial of degree d has at most $d-1$ non-repelling cycles.

2D: The Hénon map is a biholomorphism of $\mathbb{C}^2$, hence it has no critical points in the usual sense.





Differences between 1D and 2D

Different methods: Many tools from complex analysis and complex dynamics in one variable do not extend to higher dimensions.

Different phenomena: For example, unlike for 1D polynomials, the number of attracting periodic points (sinks) of a Hénon map is not bounded by the degree of the map.

Theorem (Newhouse Phenomenon)

There exist Hénon maps with infinitely many sinks, accumulating on a Smale's horseshoe.

Theorem (Coexistence Phenomena - Benedicks, Palmisano)

For every $k \geq 1$ there exists a set of parameters E_k in the parameter space of real Hénon maps such that for every $(a, b) \in E_k$ the map $f_{a,b}$ has at least k attractive periodic orbits and a strange attractor.

The foliation of U^+

Böttcher coordinate: there exists a unique holomorphic function $\varphi^+ : V^+ \to \mathbb{C} - \overline{\mathbb{D}}$ such that

- $\varphi^+ \circ H = (\varphi^+)^2$
- $\varphi^+(x, y) \sim x$, when $(x, y) \to \infty$ in V^+.

U^+ is foliated by copies of $\mathbb{C}$ given by the extended level sets of φ^+.

Critical locus

The critical locus $\mathcal{C}$ is the set of tangencies between the foliations of the escaping sets U^+ and U^-. $\mathcal{C}$ can also be described as the set of zeroes of the holomorphic function w, where

$$d \log \varphi^+ \wedge d \log \varphi^- = w(x,y) dx \wedge dy.$$

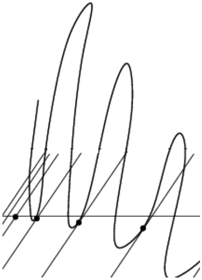


Fig.: Tangencies between leaves of two foliations

Bedford-Smillie: The set $\mathcal{C}$ is a nonempty, closed analytic subvariety of $U^+ \cap U^-$ and is invariant under the Hénon map.

Stable/unstable critical loci

Unstable Critical Locus $\mathcal{C}^u$: the set of tangencies between the foliation of U^+ and the "lamination" of J^-.

Stable Critical Locus $\mathcal{C}^s$: set of tangencies between the foliation of the escaping set U^- and the "lamination" of J^+.

J^+ and J^- are not always laminar.
Bedford-Smillie: If J^+ and J^- are both laminated and if the laminations are transversal along J then f is hyperbolic.

Theorem (Bedford, Smillie)

$$\overline{\mathcal{C}} \cap J^+ \cap U^- \neq \emptyset \text{ and } \overline{\mathcal{C}} \cap J^- \cap U^+ \neq \emptyset.$$

It is not true that $\mathcal{C}^s = \partial\mathcal{C} \cap (J^+ \cap U^-)$ and $\mathcal{C}^u = \partial\mathcal{C} \cap (J^- \cap U^+)$.

The relation between $\mathcal{C}$, $\mathcal{C}^s$ and $\mathcal{C}^u$ is "rather mysterious is general".

General questions

- General properties of the critical locus $\mathcal{C}$.
  Is the critical locus smooth or can it have singularities?
- Topological models of the critical locus $\mathcal{C}$.
- Relations between the critical loci $\mathcal{C}$, $\mathcal{C}^s$, $\mathcal{C}^u$.
- Connections between the properties of the critical locus and the dynamical properties of the map, and the connectivity of the Julia set J.
- Connections between the bifurcations of the critical locus $\mathcal{C}$ and the bifurcations of the Julia set J. Dependency of $\mathcal{C}$ on the parameters as we converge to a semi-parabolic parameter (within the regions of parabolic implosion).

Models for the critical locus I

Theorem (Lyubich, Robertson)

Let H be a Hénon map that is a small perturbation of a hyperbolic quadratic polynomial p with connected Julia set.

- There exists a unique primary component $\mathcal{C}_0$ of the *critical locus*, which is asymptotic to the x-axis. $\mathcal{C}_0$ is biholomorphic to $\mathbb{C} - \overline{\mathbb{D}}$ and it is everywhere transverse to the foliations of U^+ and U^-. Its boundary is homeomorphic to the Julia set J_p.
- All other components of $\mathcal{C}$ are iterates of $\mathcal{C}_0$ under H.

Hence the Julia set is connected & the critical locus in disconnected.

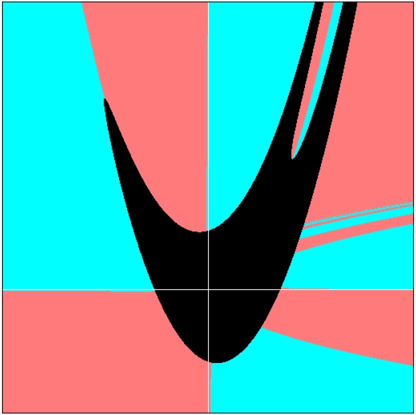


Fig.: Critical locus in the real plane. The critical locus is the boundary of the red/blue regions and we have the Lyubich-Robertson description. Picture drawn with the program Tangencies written by J. Hubbard and implemented by K. Papadantonakis at Cornell.

Application: capture the Julia set J^+

The critical locus is a powerful tool:
we used the Lyubich-Robertson critical locus as a common transverse to the foliation of U^+ and the lamination of J^+ to extend the Hubbard-Oberste-Vorth analytic structure of the escaping set U^+ to the boundary and describe the Julia set J^+:

Theorem (Tanase)

Consider complex Hénon maps that are singular perturbations of a hyperbolic polynomial with connected Julia set. The Julia set J^+ is homeomorphic to the quotient of $\mathbb{S}^1 \times \mathbb{C}$ by a discrete group of automorphisms isomorphic to $\mathbb{Z}[1/2]/\mathbb{Z}$ and an equivalence relation.

Critical locus in the horseshoe region

For any parameters (c, a) from the region

$$\text{HOV}_\beta = \left\{|c| > \beta(1 + |a|)^2\right\}, \beta \geq 2$$

the Hénon map $H_{c,a}$ is hyperbolic & the Julia set $J_{c,a}$ is a horseshoe.

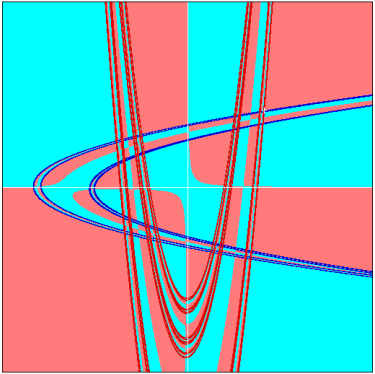


Fig.: A real horseshoe from the HOV region. The critical locus is the boundary of the red/blue regions.

Models for the critical locus II

There exists $\beta > 2$ such that the following results hold throughout the HOV$_\beta$ region:

Theorem (Firsova, R., Tanase)

The critical locus of $H_{c,a}$ is a smooth, irreducible complex analytic subvariety of $U^+ \cap U^-$, of pure dimension one.

Theorem (Firsova, R., Tanase)

The critical locus of $H_{c,a}$ is homeomorphic to a Riemann surface of infinite genus which is a countable collection of truncated spheres $(\mathcal{S}_n)_{n\in\mathbb{Z}}$ with countably many handles, glued by dynamics.

So the Julia set is disconnected & the critical locus in connected.

Theorem (Firsova, R., Tanase)

The accessible boundary of the critical locus $\mathcal{C}$ is $\mathcal{C}^s \cup \mathcal{C}^u$. The boundary of the critical locus $\mathcal{C}$ is $J^+ \cup J^-$.

Remark: The truncated sphere model was conjectured by J. Hubbard and previously proved by Firsova in perturbative setting (i.e. Hénon maps that are small perturbations of polynomials with disconnected Julia sets).

General question (perhaps conjecture): The Julia set J of the Hénon map is connected if and only if the critical locus $\mathcal{C}$ is disconnected.

Truncated sphere model

From each hemisphere we remove a countable collection of disks and a Cantor set. We further remove a point from the equator. The resulting topological object is a *truncated hemisphere*.

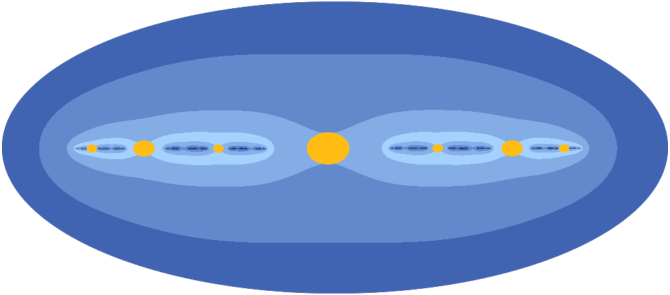


Fig.: The hemisphere of a truncated sphere.

Fig.: A truncated sphere.

Fig.: The truncated sphere $\mathcal{S}_0$ is connected to the truncated sphere $\mathcal{S}_1$ by one handle going from the northern hemisphere to the southern hemisphere.

Fig.: The truncated sphere $\mathcal{S}_0$ is connected to the truncated sphere $\mathcal{S}_2$ by two handles going from the northern hemisphere to the southern hemisphere.

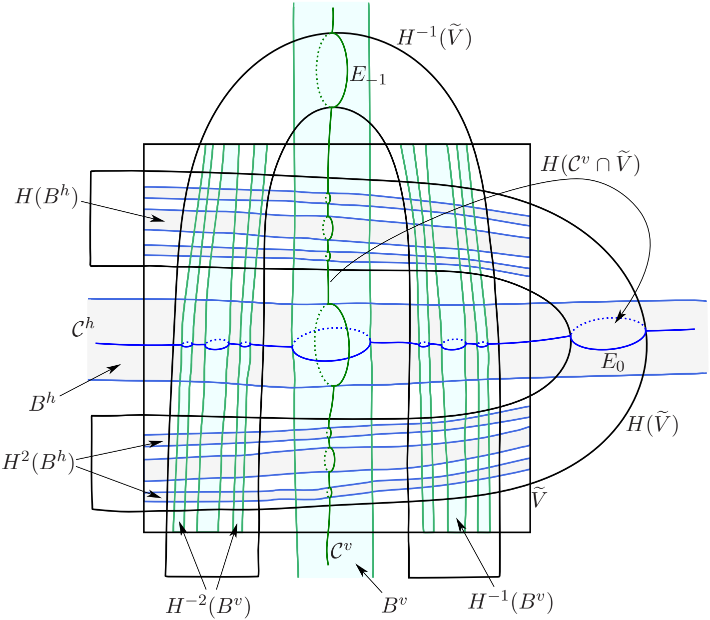


Fig.: Construction of the first truncated sphere $\mathcal{S}_0$.

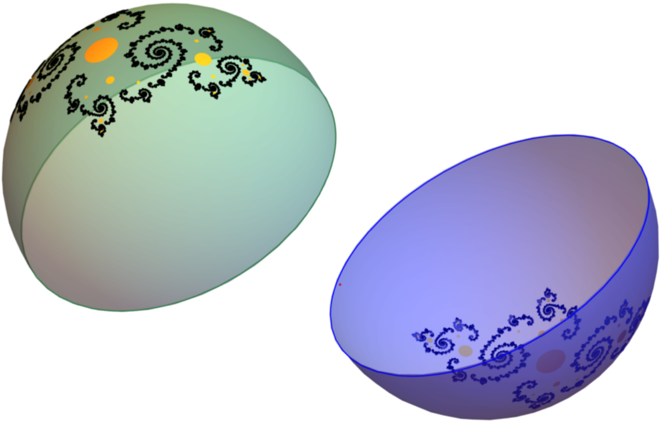


Fig.: The critical locus $\mathcal{C}^h$ on the lower hemisphere and $H(\mathcal{C}^v \cap \widetilde{V})$ on the upper hemishpere.

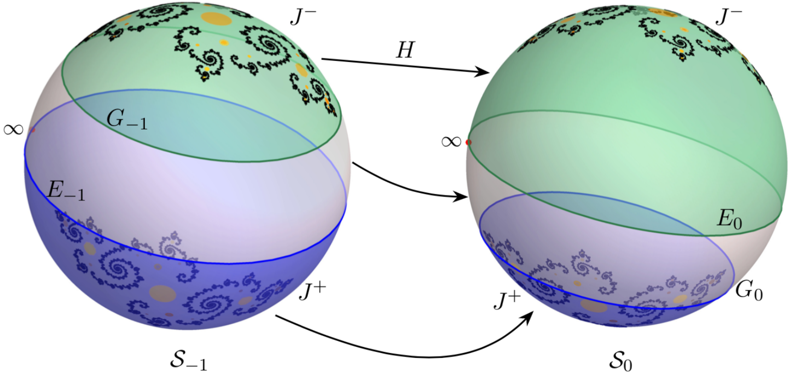
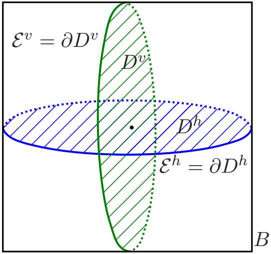


Fig.: The action of the Hénon map between two truncated spheres.

A piece of the proof: no singularities



$(\varphi^-)^2$ and $(\varphi^+)^{2^{n+1}}$ are holomorphic maps on B for some $n \geq 1$

- The function $g(x,y) := \dfrac{\partial_y (\varphi^+)^{2^{n+1}}(x,y)}{\partial_x (\varphi^+)^{2^{n+1}}(x,y)}$ is well defined and holomorphic in a neighborhood of $D^h \cup D^v \cup \partial_v(B)$ and:
  - $|g(x,y)| \leq k < 1$ on $\partial_v(B)$, since the vertical boundary of B is laminated by vertical-like leaves of the foliation of U^+.
  - $|g(x,y)| > 1$ on $\mathcal{E}^v$, since the horizontal boundary of B is laminated by horizontal-like leaves of the foliation of U^-.
- $D^h \cup \partial_v(B)$ is a distorted Hartogs figure in $\mathbb{C}^2$.

Thank you!