Geometric topology of complex domains

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Strictly pseudoconvex domains

A relatively compact connected open subset D in a complex manifold X is a strictly pseudoconvex domain if there is an open neighbourhood U of the boundary $\partial D \neq \varnothing$ and a smooth strictly plurisubharmonic function φ on U such that $D \cap U = \{\varphi < 0\}$ and $d\varphi(x) \neq 0$ for all $x \in \partial D$.

If $\dim_{\mathbb{C}} X = 1$, then every smoothly bounded domain $D \subset X$ is strictly pseudoconvex; so this case is often excluded.

The following are equivalent for a SPC domain:

(a) D is Stein.

(b) D has no compact analytic subsets of positive dimension.

(c) φ extends to a stictly plurisubharmonic function on $\overline{D}$.

Morse theory

A C^2-generic strictly plurisubharmonic function is Morse and the Morse indices of its critical points are $\le \dim_{\mathbb{C}} X$.

A *Stein* strictly pseudoconvex domain D is the interior of a finite handlebody with handles of index $\le \dim_{\mathbb{C}} D$.

$\Rightarrow \quad \mathrm{H}^p(D, \mathbb{A}) = 0$ for all $p > \dim_{\mathbb{C}} D = \frac{1}{2} \dim_{\mathbb{R}} D$

(cohomology with arbitrary constant coefficients $\mathbb{A}$)

Relative version: If $r \in (-\infty, 0)$ is a regular value of φ, then D is obtained from $D^{\le r} := \{x \in D \mid \varphi(x) \le r\}$ by attaching handles of index $\le \dim_{\mathbb{C}} D$.

$$\Rightarrow \quad \begin{cases} \mathrm{H}^p(D, \mathbb{A}) & = & \mathrm{H}^p(D^{\le r}, \mathbb{A}) & \text{for } p > \dim_{\mathbb{C}} D \\ \mathrm{H}^p(D, \mathbb{A}) & \twoheadrightarrow & \mathrm{H}^p(D^{\le r}, \mathbb{A}) & \text{for } p = \dim_{\mathbb{C}} D \end{cases}$$

Example: If $D \cong \mathbb{B}$, then $\mathrm{H}^p(D^{\le r}, \mathbb{A}) = 0$ for $p \ge \dim_{\mathbb{C}} D$.

Stein compact subsets

A compact subset $K \Subset X$ in a complex manifold is called Stein if it has a basis of Stein neighbourhoods.

Example: $K = \overline{D}$ for a Stein strictly pseudoconvex D
($\stackrel{\text{def}}{=}$ "strictly pseudoconvex Stein subset")

Beware: Stein compact subsets can be quite 'wild'. Stein neighbourhoods can be chosen strictly pseudoconvex but may become increasingly complicated.

If $K \Subset X$ is a Stein compact subset, then

$$\mathrm{H}^p(K, \mathbb{A}) = 0 \text{ for all } p > \dim_{\mathbb{C}} X$$

(Using sheaf cohomology or any other theory in which the cohomology of a compact subset is the direct limit of the cohomology of its open neighbourhoods.)

Making smoothly bounded domains Stein

Let $H \Subset X$ be the closure of a smoothly bounded domain without handles of index $> \dim_{\mathbb{C}} X$ in a complex manifold X.

Eliashberg (1990):

If $\dim_{\mathbb{C}} X \geq 3$, there exists an *isotopy* $\psi_t : X \times [0,1] \to X$ such that $\psi_0 = \mathrm{Id_X}$ and $\psi_1(H)$ is a *strictly pseudoconvex* Stein compact subset of X.

Gompf (2023):

If $\dim_{\mathbb{C}} X = 2$, there exists a *homeotopy* $\psi_t : X \times [0,1] \to X$ such that $\psi_0 = \mathrm{Id_X}$ and $\psi_1(H)$ is a Stein compact subset of X.

Gompf's compact sets have Stein interiors and no 'Nebenhülle' $\rightsquigarrow$ topological analogue of strict pseudoconvexity.

Very special case: There exists a Stein compact subset in $\mathbb{C}^2$ *homeomorphic* to a (smooth) tube around $S^2 \hookrightarrow \mathbb{C}^2$.

Smooth obstructions for $\dim_{\mathbb{C}} X = 2$

Example 1 (**Lisca & Matić** (1998), **N.** (1998, 2002))
A tube H around any smoothly embedded $S^2 \hookrightarrow \mathbb{C}^2$
is *not diffeomorphic* to a Stein compact subset in *any* X.
Reason: The core S^2 must become contractible
in any Stein neighbourhood of H.

Example 2 (**N.** (2002)):
A tube H around any smooth $\mathbb{RP}^2 \hookrightarrow \mathbb{C}^2$ with $\nu = +2$
is *not diffeomorphic* to a Stein compact subset in *any* X.
Reason: The non-trivial loop in $\mathbb{RP}^2$ must become
contractible in any Stein neighbourhood of H.

Gompf (2013):
$H \Subset X$ is isotopic to a strictly pseudoconvex Stein subset of X
$\Longleftrightarrow$ the complex structure on H is homotopic to a strictly
pseudoconvex Stein one through *almost* complex structures.

Details and methods

- Differential topology of Stein manifolds:

K. Cieliebak, Y. Eliashberg, *From Stein to Weinstein and back*. Symplectic geometry of affine complex manifolds. American Mathematical Society, Providence, RI, 2012.

F. Forstnerič, *Stein manifolds and holomorphic mappings*. The homotopy principle in complex analysis. Second edition. Springer, Cham, 2017.

- Topological 4-manifolds after Freedman ($\rightsquigarrow$ Gompf):

The disc embedding theorem. Edited by Stefan Behrens, Boldizsár Kalmár, Min Hoon Kim, Mark Powell and Arunima Ray. Oxford University Press, Oxford, 2021.

Rational convexity in $\mathbb{C}^n$

A compact subset $K \Subset \mathbb{C}^n$ is called rationally convex if for every $z \notin K$ there is a rational function R on $\mathbb{C}^n$ such that $|R(z)| > \max_{\zeta \in K} |R(\zeta)|$.

Duval & Sibony (1995):
K is rationally convex if and only if it has a basis of strictly pseudoconvex neighbourhoods $U_\varphi = \{\varphi < 0\}$ such that $dd^c\varphi$ extends to a Kähler form on $\mathbb{C}^n$.

Note: K must be a Stein compact set.

Cieliebak & Eliashberg (2015):
Let $H \Subset \mathbb{C}^n$, $\mathbf{n \geq 3}$, be the closure of a smoothly bounded domain without handles of index $> n$. Then H is ambiently isotopic to a *rationally convex* strictly pseudoconvex compact subset of $\mathbb{C}^n$.

$\leftrightsquigarrow$ Crucial new ingredient provided by **Murphy** (2012).

Rational convexity in $\mathbb{C}^2$

Examples 1 & 2 (**N. & Siegel** (2016)):
Tubes around either a totally real Klein bottle in $\mathbb{C}^2$ or an $\mathbb{RP}^2$ with one hyperbolic complex tangency in $\mathbb{C}^2$ are strictly pseudoconvex subsets which are *not* diffeomorphic to rationally convex strictly pseudoconvex subsets of $\mathbb{C}^2$.

Question 1:
Are those examples diffeomorphic to rationally convex subsets which are *not* strictly pseudoconvex?

Question 2:
Can Gompf (2023) be improved to 'rationally convex' as well, i.e. is the closure of every smoothly bounded domain without handles of index > 2 in $\mathbb{C}^2$ ambiently *homeotopic* to a rationally convex subset in $\mathbb{C}^2$?

Polynomial convexity in $\mathbb{C}^n$

A compact subset $K \Subset \mathbb{C}^n$ is called polynomially convex if for every $z \notin K$ there is a polynomial function P on $\mathbb{C}^n$ such that $|P(z)| > \max_{\zeta \in K} |P(\zeta)|$.

Oka (1953):
K is polynomially convex if and only if it has a basis of strictly pseudoconvex neighbourhoods $U_\varphi = \{\varphi < 0\}$ with φ a strictly plurisubharmonic exhaustion function on $\mathbb{C}^n$.

φ can be made $\equiv \|z\|^2$ at $\infty \Longrightarrow \mathrm{H}^p(K, \mathbb{A}) = 0$ for $p \geq n$

Cieliebak & Eliashberg (2015):
Let $H \Subset \mathbb{C}^n$, $\mathbf{n \geq 3}$, be a smoothly bounded compact subset without handles of index $> n$ and $\mathrm{H}^n(K, \mathbb{A}) = 0$ for any $\mathbb{A}$. Then H is ambiently isotopic to a *polynomially convex* strictly pseudoconvex compact subset of $\mathbb{C}^n$.

Note: H *may* have handles of index n.

Polynomial convexity in $\mathbb{C}^2$

Examples: Contractible strictly pseudoconvex subsets in $\mathbb{C}^2$ constructed by Gompf (2013) *cannot* be polynomially convex.

Mark & Tosun (2022):

A *contractible* polynomially convex strictly pseudoconvex subset $K \Subset \mathbb{C}^2$ is diffeomorphic to the 4-ball.

Sketch:

- ∂K is a homology sphere
- Oka (1953): cobordism without 3-, 4-handles from ∂K to S^3
- Gordon (1981): ∂K is simply connected
- Perelman (2003): ∂K is the 3-sphere
- Eliashberg (1991): ∂K has the standard contact structure
- Gromov (1985): K is (symplectomorphic to) the 4-ball $\square$

Question:

Is every polynomially convex strictly pseudoconvex $K \Subset \mathbb{C}^2$ diffeomorphic to the 4-ball with **only** 1-handles attached?