A variational problem in Kähler geometry

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2. General $F \to B$ $\rightsquigarrow$
$$\mathbb{P}F = \coprod_{b \in B} \mathbb{P}F_b,$$
$$\mathcal{O}_{\mathbb{P}F}(-1) = \coprod_{b \in B} \mathcal{O}_{\mathbb{P}F_b}(-1) \to \mathbb{P}F.$$

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Question: Is there a converse construction? Given metric k on $\mathcal{O}_{\mathbb{P}F}(-1)$ $\rightsquigarrow \quad$ metric k^F on F (hermitian!)?

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If for each $b \in B$

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is a hermitian ellipsoid (i.e., $\mathbb{C}$–linear image of the unit ball in $\mathbb{C}^n$), then take k^F the metric for which $(*)$ is $\{k^F < 1\}$.

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Otherwise for each $b \in B$ take the largest (by volume) hermitian ellipsoid contained in $(*)$; choose k^F whose unit ball bundle consists of these ellipsoids.

Real version:

Finding the largest inscribed ellipsoid in a convex $S \subset \mathbb{R}^N$ (Fritz John 1948).

Complex version:

Metrics on $\mathcal{O}_{\mathbb{P}_n}(-1) \to \mathbb{P}_n \quad \rightsquigarrow \quad$ hermitian norms on $\mathbb{C}^{n+1}$.

Point of talk:

Generalization of complex version to Kähler manifolds.

A variational problem on Kähler manifolds

The protagonists:

$\mathbb{P}_n \quad \rightsquigarrow \quad (M^n, \omega)$ connected compact Kähler manifold;

Metrics $\quad \rightsquigarrow \quad \{u \in C^\infty(M) : \omega_u \stackrel{\text{def}}{=} \omega + i\partial\overline{\partial}u > 0\} = \mathcal{H}$;

Ellipsoids $\quad \rightsquigarrow \quad$??

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G connected compact Lie group, acts on M by holomorphic isometries. Complexification $G^{\mathbb{C}}$ acts holomorphically:

$$G^{\mathbb{C}} \times M \ni (g, x) \mapsto gx \in M.$$

E.g.: $(M, \omega) = (\mathbb{P}_n, \omega_{\mathrm{FS}})$ and $G = \mathrm{SU}(n+1)$, $G^{\mathbb{C}} = \mathrm{SL}(n+1, \mathbb{C})$.

Definition

$u \in \mathcal{H}$ is *admissible* if $\omega_u = g^*\omega$ with some $g \in G^{\mathbb{C}}$.

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$\mathcal{H} \subset C^\infty(M)$ is open $\implies T\mathcal{H} \simeq \mathcal{H} \times C^\infty(M)$ canonically.
Mabuchi defined a closed 1–form α on $\mathcal{H}$: If $\xi \in T_u\mathcal{H} \simeq C^\infty(M)$,

$$\alpha(\xi) = \int_M \xi \omega_u^n.$$

But $\mathcal{H} \subset C^\infty(M)$ is convex, hence α is exact, and E is defined by

$$dE = \alpha, \qquad E(0) = 0.$$

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The problem: Given $v_0 \in \mathcal{H}$, find/characterize the minimizer in

$$\text{(P)} \qquad \min\{E(v) : v \in \mathcal{H} \text{ admissible}, \ v \geqslant v_0\}.$$

The moment map

Let $X \in \mathfrak{g}$ (Lie algebra of G). Then $g_t = \exp tX \in G$ $(t \in \mathbb{R}) \longleftrightarrow$ 1–parameter subgroup of diffeomorphisms $\longleftrightarrow$ flow of a vector field on M, denoted X_M. (WLOG: $X_M = 0 \Longrightarrow X = 0$.) Now

$$g_t^*\omega = \omega \quad \Longrightarrow \quad 0 = \mathcal{L}_{X_M}\omega = d\iota_{X_M}\omega + \overbrace{\iota_{X_M} d\omega}^{0}.$$

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Suppose $\forall X \in \mathfrak{g}$ the 1–form $\iota_{X_M}\omega$ is even exact: $\iota_{X_M}\omega = dh_X$, with $\int_M h_X \omega^n = 0$. As $\mathfrak{g} \ni X \mapsto h_X(x) \in \mathbb{R}$ is linear $\forall x \in M \Longrightarrow \exists \mu : M \to \mathfrak{g}^*$ such that $\langle \mu, X \rangle = h_X$.

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Example: $(M, \omega) = (\mathbb{P}_n, \omega_{\mathrm{FS}})$, $G = \mathrm{SU}(n+1) \Longrightarrow$

$$\mathfrak{g} = \{X \in \mathrm{Mat}^{(n+1)\times(n+1)} : X^\dagger = -X, \operatorname{tr} X = 0\}$$

$$\langle \mu(x), X \rangle = i\frac{x^\dagger X x}{x^\dagger x}, \qquad x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}.$$

The problem was: Given $v_0 \in \mathcal{H}$,

$$\text{(P)} \qquad \min\{E(v) : v \in \mathcal{H} \text{ admissible}, \ v \geqslant v_0\}.$$

If $u \in \mathcal{H}$, let $C_u = \{x \in M : u(x) = v_0(x)\}$.

Theorem (1)

(P) has a minimizer. Let $u \geqslant v_0$ be admissible: $\omega_u = g^*\omega$, with $g \in G^{\mathbb{C}}$. This u is minimizer in (P) $\iff$ $0 \in$ convex hull of $\mu(gC_u) \subset \mathfrak{g}^*$.

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If $X \in \mathfrak{g}$, let $N_X = \{x \in M : \langle \mu, X \rangle$ vanishes to 2nd order at $x\}$.

Theorem (2)

Suppose $\forall X \in \mathfrak{g} \setminus \{0\}$, $0 \notin$ convex hull of $\mu(N_X) \subset \mathfrak{g}^*$. Then (P) has unique minimizer.

(If $\exists X \in \mathfrak{g} \setminus \{0\}$ such that $0 \in$ convex hull of $\mu(N_X)$, then for some v_0 the minimizer is not unique.)

Theorem (1, special case: $u = 0$, $g = \mathrm{id}$)

0 minimizes $E(v)$ over admissible $v \geqslant v_0 \iff 0 \in$ convex hull of $\mu(C_0)$.

Idea of proof, necessity ($\Rightarrow$): If $0 \notin$ convex hull of $\mu(C_0) \Longrightarrow \exists X \in \mathfrak{g}$ such that $\langle \mu, X \rangle > 0$ on C_0 $\rightsquigarrow$ get variation $u_t \geqslant v_0$ such that $E(u_t) < E(0)$ for $0 < t < \varepsilon$, contradiction.

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Sufficiency ($\Leftarrow$): Convex hull condition $\overset{\text{approx}}{\iff}$ some derivative $= 0 \overset{?}{\implies}$ minimum? Yes, if function involved is convex.

So this part of the proof depends on intrinsic convexity in $\mathcal{H}$, framed in terms of Mabuchi's Riemannian metric on $\mathcal{H}$: If $\xi, \eta \in T_u\mathcal{H} \simeq C^\infty(M)$,

$$g_{\text{Mabuchi}}(\xi, \eta) = \int_M \xi \eta \, \omega_u^n.$$

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0 *minimizes* $E(v)$ *over admissible* $v \geqslant v_0 \iff 0 \in$ *convex hull of* $\mu(C_0)$.

Proof of sufficiency ($\Leftarrow$): Convex hull condition $\iff$ $\exists$ probability measure m on C_0 such that $\int_{C_0} \mu \, dm = 0$.

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Suppose $v \geqslant v_0$ is admissible: $\omega_v = \gamma^* \omega$, with $\gamma \in G^{\mathbb{C}}$. WLOG $\gamma = \exp iX$, $X \in \mathfrak{g}$. Connect $u = 0$ and v by geodesic in Mabuchi's metric.

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Lemma

$\psi(t) = 2 \int_0^t (\exp i\tau X)^* \langle \mu, X \rangle \, d\tau + tE(v)$, $0 \leqslant t \leqslant 1$, is geodesic in $\mathcal{H}$, connects $u = 0$ and v, and $\psi(t) \geqslant v_0$ $\forall t$.

If $x \in C_0$, then $0 = v_0(x) \leqslant \psi(t)(x)$; and $=$ holds for $t = 0$. Hence $0 \leqslant \dot\psi(0)(x)$ and

$$E(0) = 0 \leqslant \int_{C_0} \dot\psi(0) \, dm = \int_{C_0} \big(2\langle \mu, X \rangle + E(v)\big) \, dm = 0 + E(v).$$

Summary:

We considered a Kähler analog of a variational problem, posed by Fritz John in convex geometry. We formulated theorems concerning the existence, uniqueness, and characterization of a solution. The Kähler problem involved a group action; our results were framed in terms of the associated moment map.

The proofs use the geometry of the space of Kähler potentials, first introduced by Mabuchi.