Length spectral rigidity of expanding circle maps

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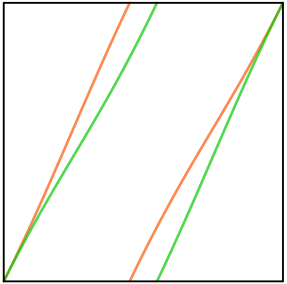
(Based on joint work in progress with Vadim Kaloshin)

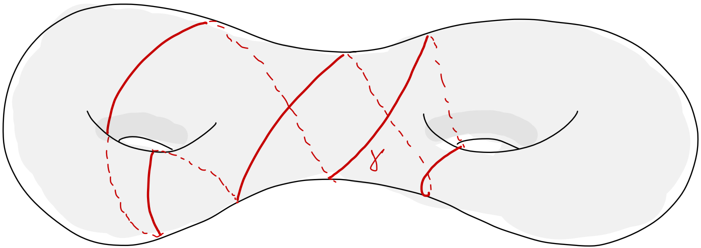


Complex Analysis, Geometry, and Dynamics III

June 13, 2024

Surfaces and manifolds of negative curvature

M closed orientable surface of genus $d \geqslant 2$.
g complete Riemannian metric of negative (Gaussian) curvature $K(g) < 0$



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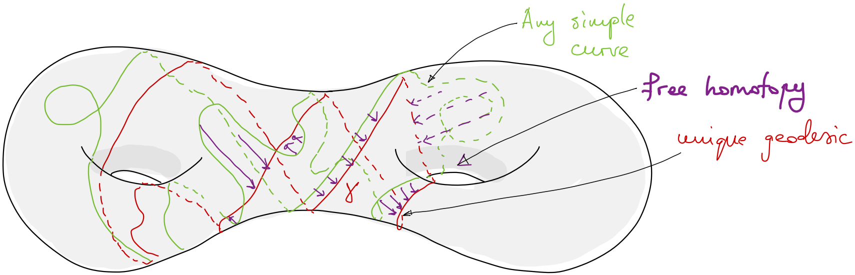

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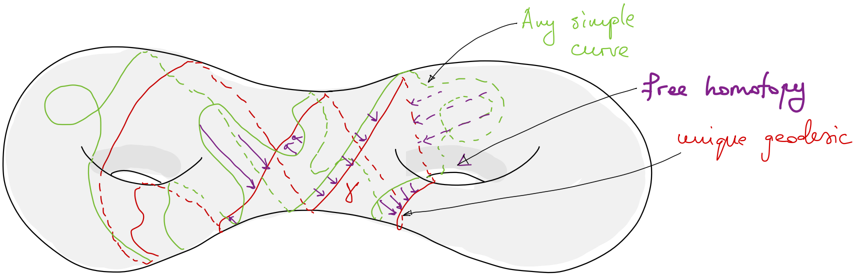

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The same holds if (M, g) is a closed Riemannian manifold of negative (sectional) curvature $sec(g) < 0$.

Length spectrum for metrics

$\mathcal{C}$ set of all free-homotopy classes of closed curves on (M, g).
$\forall c \in \mathcal{C}$, let γ_c be the unique geodesic in the class c.

$\mathcal{L}_g := \{\lvert\gamma_c\rvert_g : c \in \mathcal{C}\}$	length spectrum
$\mathcal{ML}_g : \mathcal{C} \ni c \mapsto \lvert\gamma_c\rvert_g$	marked length spectrum

Main Rigidity Question

Does (marked) length spectrum define the metric uniquely?

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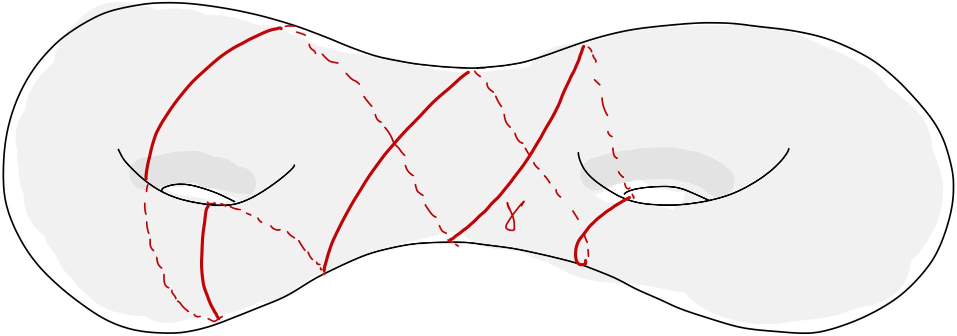
Why do we expect length spectral rigidity (morally)?

(M, g), $dim(M) = 2$, $K(g) = -1$, $|\gamma|_g = 10$ $\quad \cdot\, diam(M, g)$



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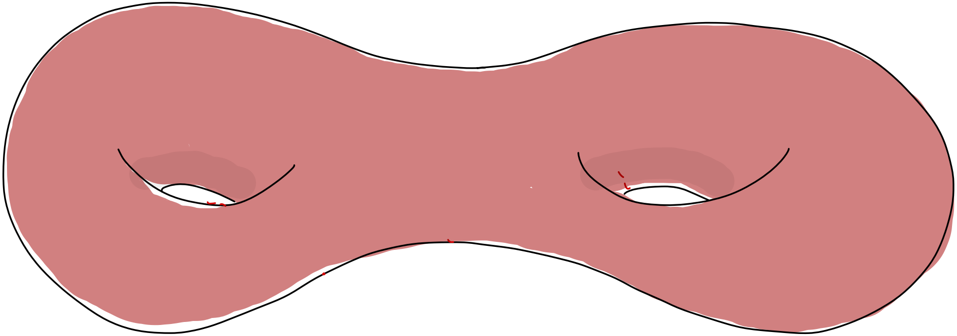

Exponentially dense!

Marked length spectral rigidity for metrics

Conjecture (Burns–Katok'85)

For $\dim(M) \geqslant 2$ *and a pair* (M, g_1), (M, g_2) *of negatively curved Riemannian metrics on* M*,*

$$\text{if} \quad \mathcal{ML}_{g_1} = \mathcal{ML}_{g_2}, \quad \text{then} \quad g_1 = g_2 \quad \textit{(up to change of coordinates)}$$

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- $\dim(M) = 2$: **Otal**'90 (using theory of geodesic flows), **Croke**'90; **Guillarmou–Lefeuvre**'19, **Guillarmou–Lefeuvre–Paternain**'23 (for Anosov geodesic flows)
- $\dim(M) > 2$: **Katok**'88 (for fixed conformal classes, using ergodic theory), **Hamenstädt**'99 (for locally symmetric spaces, using rigidity of entropy due to **Besson–Courtois–Gallot**'95),

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Theorem (Guillarmou–Lefeuvre'19)

Let (M, g) be a closed Riemannian manifold of dimension $n \geqslant 3$ with sec$(g) < 0$. Then there exist $\varepsilon > 0$, $N = N(n)$ so that if g_0 is another smooth metric on M with sec$(g_0) < 0$ and such that

$$\mathcal{ML}_{g_0} = \mathcal{ML}_g, \quad \|g_0 - g\|_{C^N(M)} < \varepsilon, \quad \text{then} \quad g_0 = g \quad \text{(up to change of coordinates).}$$

Length spectral rigidity for metrics ($\mathcal{L}_{g_1} = \mathcal{L}_{g_2} \overset{?}{\Rightarrow} g_1 = g_2$)

In general, this is not true (examples of **Sunada** and **Vigneras**).

However, the following local length spectral rigidity question (analogous to Guillarmou–Lefeuvre's result) is **completely open**!

Question/Conjecture (Sarnak)

Are metrics of negative curvature locally rigid with respect to their length spectra?

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I.e., given (M, g) with sec$(g) < 0$, there exist $\varepsilon > 0$ and $N > 0$ such that if g_0 is another smooth metric on M with sec$(g_0) < 0$ and such that

$$\mathcal{L}_{g_0} = \mathcal{L}_g, \quad \|g_0 - g\|_{C^N(M)} < \varepsilon, \quad \textit{then} \quad g_0 = g \quad \textit{(up to change of coordinates).}$$

... Related to Laplace spectral rigidity question (Kac's famous 'Can you hear the shape of the drum?')

A simpler question?

Moral 2-step reduction

Geodesic flows (3-dimensional) Periodic trajectories = closed geodesics	$\longrightarrow$	Anosov diffeomorphisms (2-dimensional) Periodic points	$\longrightarrow$	Expanding maps on the circle (1-dimensional) Periodic points

Expanding circle maps

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$

For today, $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ is **expanding circle map** if f is a C^s-smooth ($s > 1$) degree $d \geqslant 2$ orientation preserving covering of $\mathbb{S}^1$ such that $f'(x) \geqslant \Lambda > 1$, $\forall x \in \mathbb{S}^1$, normalized as $f(0) = 0$.

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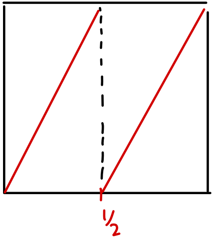
Examples:

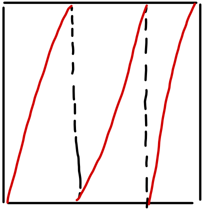
$F_d : x \mapsto dx \mod 1$ (degree $d \in \mathbb{N}$, $d \geqslant 2$)

$x \mapsto 3x + \sin(2\pi x) \mod 1$



degree 2

F_2



degree 3

Combinatorics of expanding circle maps

Theorem (Shub'69)

Every degree $d \geqslant 2$ expanding circle map f is topologically conjugate to $F_d \colon x \mapsto dx \mod 1$ via an orientation-preserving homeomorphism $\varphi \colon \mathbb{S}^1 \to \mathbb{S}^1$, $\varphi(0) = 0$:

$$\varphi^{-1} \circ f \circ \varphi = F_d$$

Up to a topological change of coordinates, all expanding circle maps of degree d are the same $\Rightarrow$ all maps have the same combinatorics of periodic orbits!

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Theorem (Sullivan'88)

φ is quasisymmetric, and hence Hölder continuous.

It follows that any two expanding circle maps f, g of the same degree are topologically conjugate via some quasisymmetric homeomorphism ψ. However, ψ cannot be better than that (usually, it is nowhere differentiable)!

Periodic cycles and length spectrum

$F_d \colon x \to dx \mod 1$

- $C \subset \mathbb{S}^1$ is a **periodic cycle**, with period $per(C) = n$, if $|C| = n$, $F_d(C) = C$, and $F_d|_C$ is a cyclic permutation.
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- $\mathcal{C}$ = the set of all periodic cycles of F_d.

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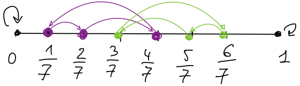

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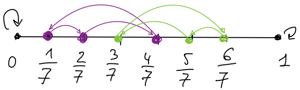

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- for $C \in \mathcal{C}$, $\varphi(C)$ is the **corresponding** periodic cycle for f, $per(\varphi(C)) = per(C)$.
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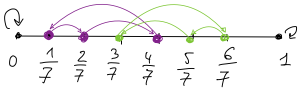

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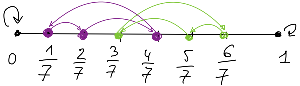

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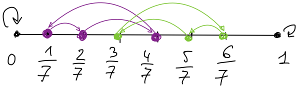

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Note: a) $\mathcal{L}_{F_d} = \{n \cdot \log d\colon n \in \mathbb{N}\}$; b) if $h\colon \mathbb{S}^1 \to \mathbb{S}^1$ is a diffeomorphism, then $\mathcal{L}_f = \mathcal{L}_{h^{-1} \circ f \circ h}$.

Why ‘length’ spectrum? (A digression, due to McMullen)

$\mathcal{B}_d = \left\{ f(z) = z \prod_1^{d-1} \frac{z - a_i}{1 - \overline{a_i} z} : \lvert a_i \rvert < 1 \right\}$ $\longrightarrow \forall f \in \mathcal{B}_d$, $f\lvert_{\mathbb{S}^1}$ is a degree $d \geqslant 2$ expanding circle map	Hyperbolic surfaces $X = (S, g)$ of genus $d > 1$ (with curvature $K(g) = -1$)
All periodic cycles with $\lvert C \rvert_f < \log 2$ are simple and ‘disjoint’	All closed geodesics with $\lvert\gamma\rvert_g < \log(3 + 2\sqrt{2})$ are simple and disjoint
$\exists$ simple cycle C with $\lvert C \rvert_f = O(d)$	$\exists$ simple closed geodesic γ with $\lvert\gamma\rvert_g = O(\log d)$
If $(C_i)_1^n$ is a binding collection of cycles, then $\forall M$, the closure of $\left\{ f \in \mathcal{B}_d : \sum_1^n \lvert C_i \rvert_f \leqslant M \right\}$ is cpt in the moduli space of degree d rational maps	If $(\gamma_i)_1^n$ is a binding collection of closed geodesics, then $\forall M$, $\left\{ X = (S, g) \in Teich(S) : \sum_1^n \lvert\gamma_i\rvert_g \leqslant M \right\}$ is compact in $Teich(S)$

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$$\text{If } \mathcal{ML}_f = \mathcal{ML}_g, \quad \text{then} \quad f = g \quad \text{up to a } C^s\text{-smooth change of coordinates}$$

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What about (unmarked) length spectrum?

Length spectral rigidity for expanding circle maps: negative result

$$\mathcal{L}_{f,n} = \{|C|_f \colon C \in \mathcal{C}, per(C) = n\} \quad \text{length spectrum for period } n$$

Theorem (D.-Kaloshin'23)

Given $\varepsilon > 0$, $d \in \mathbb{N}$, $d \geqslant 2$ and $s > 1$,
there exist C^s-smooth non-linear expanding circle maps $f, g \colon \mathbb{S}^1 \to \mathbb{S}^1$ of degree d such that

- $\mathcal{L}_{f,n} = \mathcal{L}_{f,n}$, $\forall n \in \mathbb{N}$,
- $\|f - g\|_{C^s} \leqslant \varepsilon$,

but f and g are **not** *conjugate by any orientation-preserving diffeomorphism.*

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Are expanding circle maps locally rigid with respect to their length spectra?

Length spectral rigidity for expanding circle maps: positive result

Conjecture

Let $g\colon \mathbb{S}^1 \to \mathbb{S}^1$ *be a* C^s*-smooth,* $s > 1$*, expanding circle map of degree* $d \geqslant 2$*. Then there exists* $\varepsilon = \varepsilon(g)$ *such that: If* f *is another such map with*

$$\|f - g\|_{C^s} \leqslant \varepsilon, \quad \mathcal{L}_{f,n} = \mathcal{L}_{g,n} \quad \forall n \in \mathbb{N},$$

then f *and* g *are* C^s*-conjugate.*

We establish this conjecture under some additional assumption:

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We establish this conjecture under some additional assumption: The length spectrum $\mathcal{L}_f$ is β-**sparse** if

$$\exists \beta > 0, A > 0 \text{ such that } \quad \big||C|_f - |C'|_f\big| \geqslant A d^{-\beta \cdot n} \quad \forall \text{ cycles } C \neq C' \text{ with } per(C) = per(C') = n.$$

[**McMullen**'24] Smooth β-sparse maps exist. (A similar condition was studied for hyperbolic metrics by **Dolgopyat–Jakobson**)

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Theorem (D.-Kaloshin'24)

$\exists \beta_0 > 0$ such that the conjecture above is true for all g with β-sparse spectrum with $\beta < \beta_0$.

Negative result: idea of proof

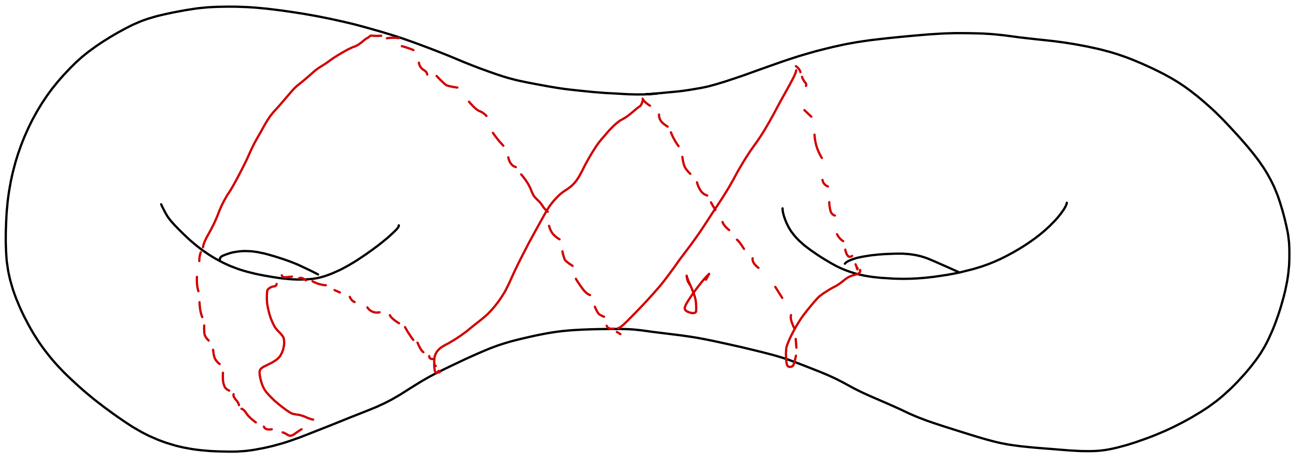
Positive result: idea of proof

Thank you for your attention!

Surfaces and manifolds of negative curvature

S closed orientable surface of genus $g \geq 2$

g complete metric of negative curvature : $K(g) < 0$



(S, g)

γ closed geodesic on (S, g) (can self-intersect)

$|\gamma|_g$ length of γ in metric g

> How do we account for all closed geodesics?

Surfaces and manifolds of negative curvature

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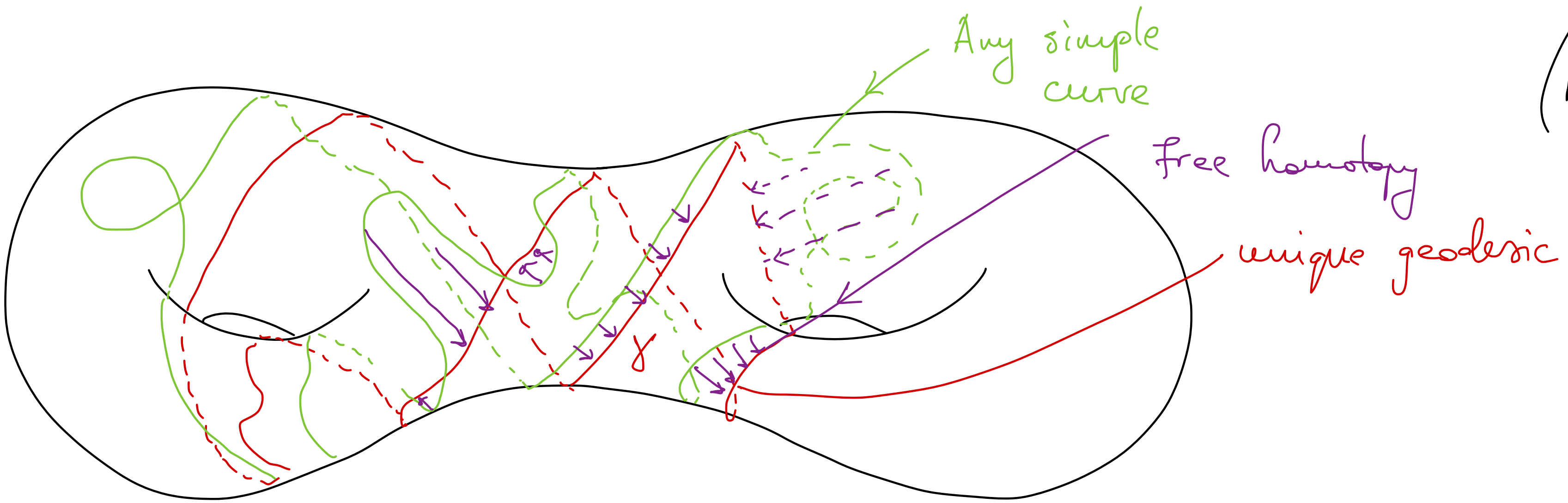
(M, g)



γ closed geodesic on (M, g) (can self-intersect)

$|\gamma|_g$ length of γ in metric g

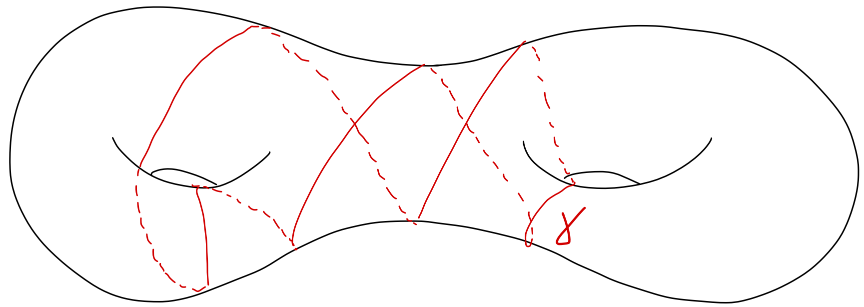
How do we account for all closed geodesics?

$K(g) < 0 \Rightarrow$ Any closed curve can be free homotoped to a unique geodesic

Similarly, if (M, g) is a closed Riemannian manifold w/ $\sec(g) < 0$

Length spectra for metrics

$\mathcal{C}$ = set of free homotopy classes of closed curves on (M,g)



(M,g)

$\forall c \in \mathcal{C}$, γ_c is the unique geodesic in the class c

$\mathcal{L}_g = \{ |\gamma_c|_g : c \in \mathcal{C} \}$ length spectrum

$ML_g : c \mapsto |\gamma_c|_g$, $c \in \mathcal{C}$ marked length spectrum

Main Question (Rigidity)

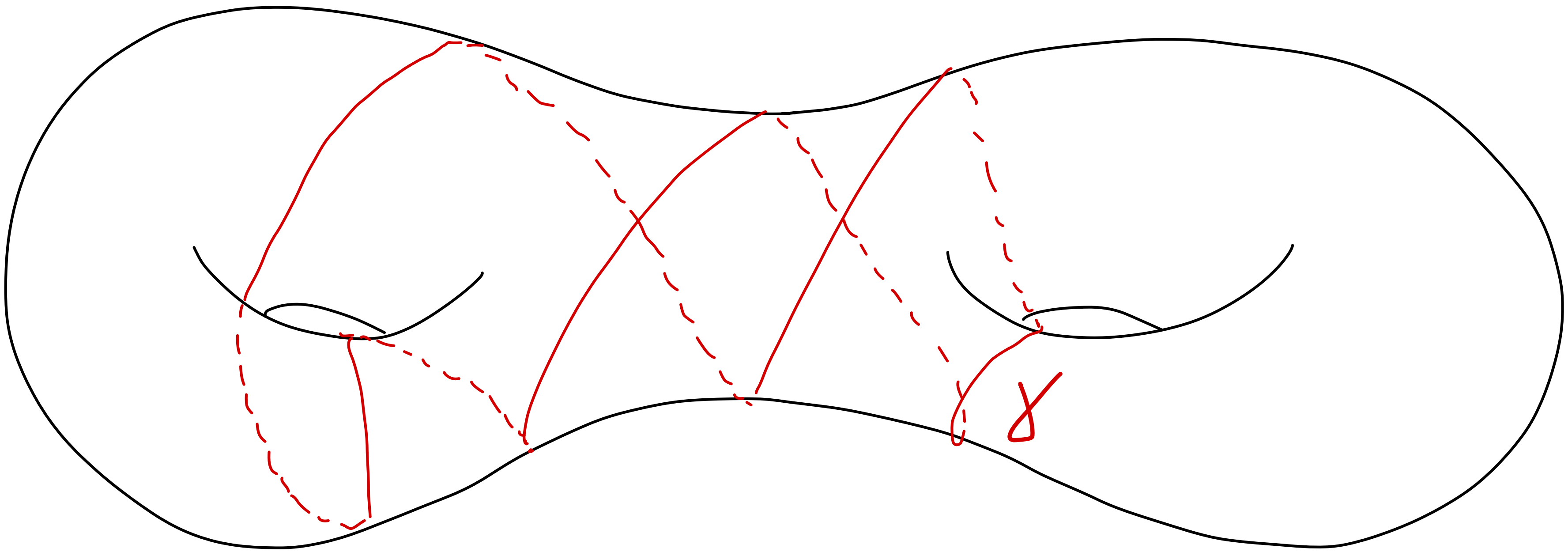
Does (marked) length spectrum define the metric uniquely?

$(M,g_1), (M,g_2)$:
- $ML_{g_1} = ML_{g_2}$
- $L_{g_1} = L_{g_2}$

$\overset{?}{\Longrightarrow}$ $g_1 \simeq g_2$ (up to change of coord, $\exists\, \varphi : M \to M$ diffeo s.t. $\varphi^* g_1 = g_2$)

Why do we expect rigidity? (morally)

Check isodiametric



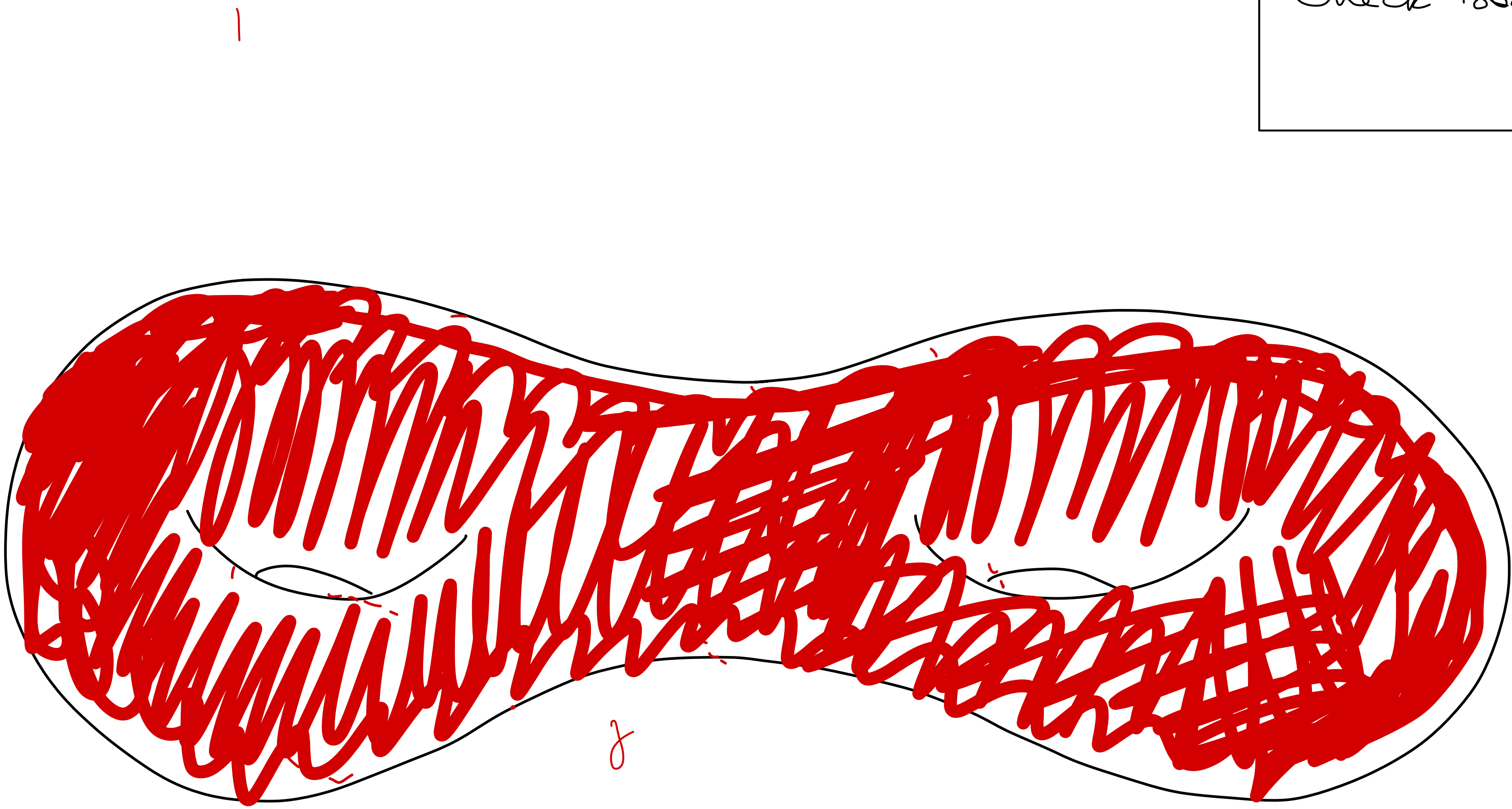
(M, g) $\dim M = 2$

$\operatorname{diam} M = 1$

$|\gamma|_g = 10$

Why do we expect rigidity? (morally)

Check isodiametric



(M, g) $\dim M = 2$

$\operatorname{diam} M = 1$

$|\gamma|_g = 10^{10^{10}}$ equidistribution of closed geodesics.

Marked Length Spectral Rigidity ($ML_{g_1} = ML_{g_2} \overset{?}{\Rightarrow} g_1 \simeq g_2$)

Conjecture (Burns - Katok): $\dim M \geq 2$

$$ML_{g_1} = ML_{g_2} \Rightarrow g_1 \simeq g_2$$

Check again
homotop. to identity

Wide open

A sample
History

+ instruments

Rigidity for geodesic flows

Thm (Guillarmou - Lefeuvre '19) (Local rigidity)

$$\left.\begin{array}{l} ML_{g_1} = ML_{g_2} \\ |g_1 - g_2|_{C^N} \leq \varepsilon \end{array}\right\} \Rightarrow g_1 \simeq g_2$$

+ instruments

Length Spectral Rigidity $\left(L_{g_1} = L_{g_2} \overset{?}{\Longrightarrow} g_1 \simeq g_2 \right)$

In general, the answer is NO

(Examples of Sunada, Vigneras) : the counter examples are global

Check

Question (Sarnach) : [Local Length Spectral Rigidity]
Conjecture

Are metrics of negative curvature locally rigid w.r.t. their length spectra?

I.e. $\forall\ (M, g_1)$, $\sec(g_1) < 0$ $\exists\ \varepsilon = \varepsilon(M, g_1)$

s.t. if g_2 w/ $\sec(g_2) < 0$

$$\begin{matrix} L_{g_1} = L_{g_2} \\ |g_1 - g_2| < \varepsilon \end{matrix} \overset{?}{\Longrightarrow} g_1 \simeq g_2$$

Completely open! Related to Laplace spectral rigidity question 'Can you hear the shape of the drum'

Moral 2-step reduction

A simplier question?

Poincaré section

'Unstable mfd'

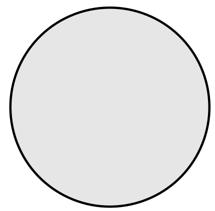
Geodesic flow $\longrightarrow$ Anosov diffeo $\longrightarrow$ Expanding map of the circle

3-dim

2-dim

Amy Wilkinson pic

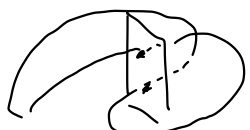
Cat Map



$x \mapsto 2x$ mod 1

Geodesics

Periodic points



$\mathbb{S}^1/\mathbb{Z}$

Periodic points

Expanding circle maps

$\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$

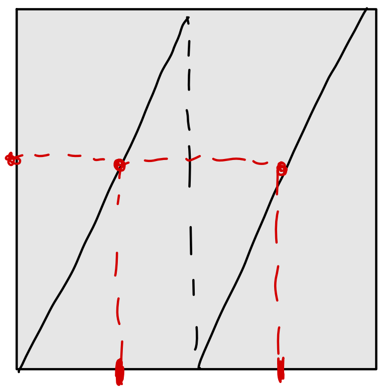
A simplest example of hyperbolic dyn. syst.

- A C^r- smooth, $r \geq 1$, orienta circle endomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$ is expanding if $f'(x) \geq \Lambda > 1$ $\forall x \in \mathbb{S}^1$.
- $f|_{\mathbb{S}^1}$ is covering, and hence have a degree, $d \geq 2$

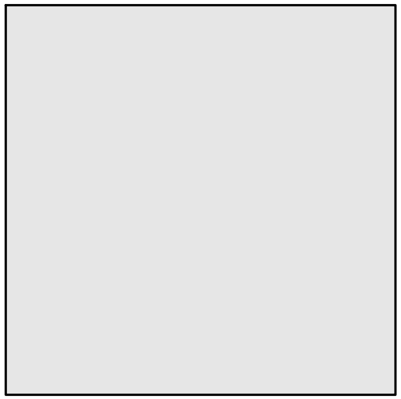
Examples:

$F_d: x \mapsto d\,x \mod 1$ (degree $d \in \mathbb{N}$, $d > 2$) $\qquad x \mapsto 3x + \sin \pi x \mod 1$

F_2



degree 2



degree 3

Expanding circle maps: periodic cycles and length spectrum

Fact: Every degree $d \geq 2$ expanding circle map f is topologically conjugate to $F_d : x \mapsto dx \bmod 1$ via some orientation-preserving homeomorphism $\varphi : S^1 \to S^1$

i.e. $$\varphi^{-1} \circ f \circ \varphi = F_d$$

Up to topological changes of coordinates, all expanding maps of the circle are the same! $\Longrightarrow$ have the same combinatorics.

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Up to topological changes of coordinates, all expanding maps of the circle are the same! $\Longrightarrow$ have the same combinatorics.

F_d: $C \subset \mathbb{S}^1$ is a periodic cycle, w/period n, if $|C| = n$,

$F_d(C) = C$, $F|C$ cyclic permutation

$x \in C$ has period n $\Longleftrightarrow$ $x = \dfrac{p}{d^n - 1}$, $p \in \mathbb{N}$

$\mathrm{per}(x) = \mathrm{per}(C) = n$

Example

$d = 2$, $n = 3^{10}$

Exponentially dense

Expanding circle maps: periodic cycles and length spectrum

Fact: Every degree $d \geq 2$ expanding circle map f is topologically conjugate to $F_d : x \mapsto dx \mod 1$ via some orientation-preserving homeomorphism $\varphi : S^1 \circlearrowleft$, $\varphi(0) = 0$

i.e. $\varphi^{-1} \circ f \circ \varphi = F_d$

Up to topological changes of coordinates, all expanding maps of the circle are the same! $\Rightarrow$ have the same combinatorics.

Fact [Sullivan]: φ is quasisymmetric, and in particular Hölder

Weiershtras

However, in general, cannot be better than that!

(nowhere differentiable!)

Thm: If φ is differentiable at at least 1 point, then φ is smooth (and hence $F_d'(0) = f'(0)$, which need not be the case)

Ref

Expanding circle maps: periodic cycles and length spectrum

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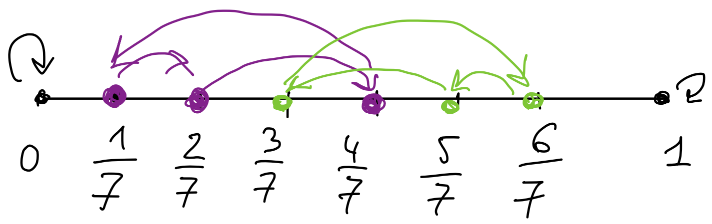

$d = 2$, $\quad n = 3 \quad\quad 2^3 - 1 = 7$

Expanding circle maps: periodic cycles and length spectrum

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$F_d(C) = C$, $F|C$ cyclic permutation

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$\mathrm{per}(x) = \mathrm{per}(C) = n$

Example

$d = 2$, $n = 3^{10}$

Exponentially dense

$\mathcal{C}$ = the set of all periodic cycles of F_d

$\Rightarrow$ $\varphi(C)$ is the corresponding periodic cycle for f, $\mathrm{per}(\varphi(C)) = n$

$$|C|_f := \sum_{x \in C} \log f'(\varphi(x))$$ length of C_f ($e^{|C|_f}$ is the multiplier $(f^n)'(x)$, $x \in \varphi(C)$)

$$\mathcal{L}_f = \{ |C|_f : C \in \mathcal{C} \}$$ length spectrum

$\boxed{\mathcal{L}_{F_d} = \{\log d, 2\log d, 3\log d, \dots\}}$

$h \circ f^n \circ h^{-1} = g^n$

$\dfrac{h'(f \circ h^{-1})}{h'(h^{-1})} \cdot f'(h^{-1}) = g'$

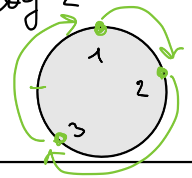
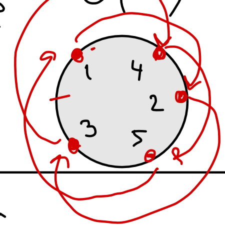
$$\mathcal{ML}_f : C \longmapsto |C|_f$$ marked length spectrum

Note: Smooth conjugacies preserve multipliers $\Rightarrow$ preserve $\mathcal{ML}_f = \mathcal{ML}_{h^{-1} \circ f \circ h}$

Why 'length' spectrum? (Digression)

There are several motiv.

C. McMullen showed the following analogy to lengths of simple closed geodesics on hyperbolic surfaces

$\mathcal{B}_d = \{ f(z) = z \prod_1^{d-1} \frac{z-a_i}{1-\bar{a}_i z} : \mathbb{D} \to \mathbb{D} \mid \lvert a_i \rvert < 1 \}$ $\to \forall f \in \mathcal{B}_d$, $f\vert_{S^1}$ is a degree d expanding circle map	Hyperbolic surfaces $X = (S, g)$ of genus $d > 1$ $(K(g) = -1)$
All periodic cycles w/ $\lvert C \rvert_f < \log 2$ are simple and 'disjoint' 	All closed geodesic $\lvert \gamma \rvert < \log(3 + 2\sqrt{2})$ are simple and disjoint
$\exists$ simple cycle w/ $\lvert C \rvert_f = O(d)$ 	$\exists$ simple closed geodesic γ with $\lvert \gamma \rvert = O(\log g)$
Let $(C_i)_1^n$ is a binding collection of cycles. $\forall M$ $\{ f \in \mathcal{B}_d \mid \sum_i^n \lvert C_i \rvert_f \leq M \}$ cpt in the moduli space of rational maps	Let $(\gamma_i)_1^n$ binding collection of closed curves, then $\forall M$ $\{ X \in \mathrm{Teich}(S) \mid \sum \lvert \gamma_i \rvert_X \leq M \}$ compact in $\mathrm{Teich}(S)$

Length spectral rigidity for expanding circle maps

Main Rigidity Question:

Does (marked) length spectrum define the expanding circle map uniquely?

Thm: [Shub - Sullivan '85] f, g C^γ - smooth ($\gamma > 1$; $\gamma = \infty$; $\gamma = \omega$) expanding degree $d \geq 2$.

$$\mathcal{ML}_f = \mathcal{ML}_g \Rightarrow f \simeq g$$ (up to C^γ-smooth change of coordinates, i.e., $\exists\, \psi$ C^γ smooth s.t. $\psi^{-1} \circ f \circ \psi = g$)

Generalizations by de la Llave, Marco, Moriyón, Gogolev, Kalinin, Sadovskaya.

What about unmarked length spectral rigidity?

$\rightsquigarrow$ In general, the answer is $\boxed{\text{no}}$

Unmarked L.S.R. : Negative results

$$\mathcal{L}_{f,n} = \{ |C|_f \mid C \in \mathcal{C},\ per(C) = n \}$$ length spectrum for period n

$$\mathcal{L}_f = \bigcup_{n \in \mathbb{N}} \mathcal{L}_{f,n}$$

Thm (D-Kaloshin):

Given $\varepsilon > 0$, $d \in \mathbb{N}$, $d \geq 2$ and $\gamma > 1$

$\exists$ $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ C^γ-smooth non-linear expanding circle maps of degree d s.t.

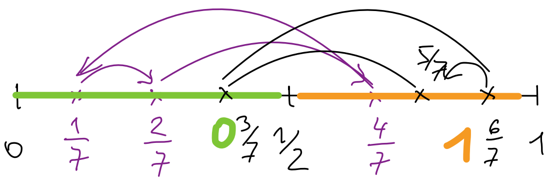
- $\mathcal{L}_{f,n} = \mathcal{L}_{g,n}$ $\forall n \in \mathbb{N}$
- $\| f - g \|_{C^\gamma} \leq \varepsilon$

but f and g are not smoothly conjugate by an orientation-preserving homeo.

Unmarked L.S.R.: Negative results (idea of proof)

Assume $d=2$ (for simplicity)

<u>Recall:</u> For F_2, every $x \in C$, $\mathrm{per}(C) > 1$, can be coded using 0's and 1's



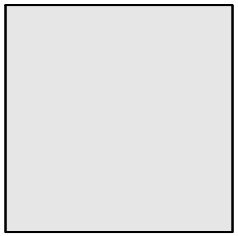
$\frac{1}{7} = (001)$, $\frac{6}{7} = (110)$

$C = [001]$ $\quad \bar{C} = [110]$ (all cyclic permutation)

$\rightarrow$ by topological conjugacy, the same is true for any expanding circle map of degree 2.

Proof by example:

$f(x) = 2x + 0.1 \cdot \sin^2(\pi x)$ $\qquad$ $g(x) = 2x - 0.1 \sin^2(\pi x)$



$|[001]|_f \cong 2.31$ $\qquad$ $|[000]|_f = 2$ $\qquad$ $|[001]|_g \simeq 1.9$ $\qquad$ $|[000]|_g = 2$

$|[110]|_f \simeq 1.9$ $\qquad$ $|[110]|_g \simeq 2.31$

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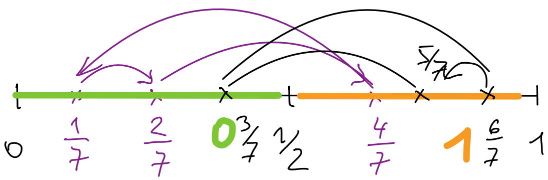

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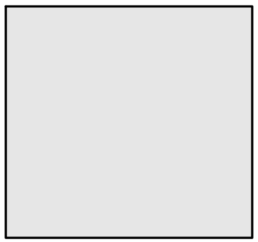
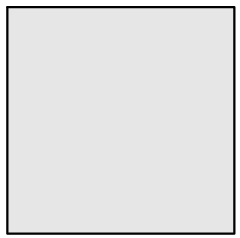


$$\boxed{g(x) = -f(-x)}$$



$g = r \circ f \circ r^{-1}$

$r(x) = -x$

$L_{f,n} = L_{g,n} \quad \forall n$ because smooth conjugacies (orient. preserving or not) preserve length spectrum

$|[001]|_f \cong 2.31$ $\quad |[000]|_f = 2$ $\quad |[001]|_g \simeq 1.9$ $\quad |[000]|_g = 2$

$|[110]|_f \simeq 1.9$ $\quad |[110]|_g \simeq 2.31$

Not smoothly conjugate by an orientation preserving homeo.

Unmarked L.S.R.: Negative results (idea of proof)

Assume $\underline{d=2}$ for simplicity

Pick f (in the class described, non-linear)

- $\|f - F_2\|_{C^\gamma} \leq \frac{\varepsilon}{2}$ (small)
- $-f(x) \neq f(-x)$

Take $r : x \mapsto -x$ and define $g := r \circ f \circ r^{-1}$

In this way

$$f \neq g$$

$$\|f-g\|_{C^\gamma} \leq \varepsilon$$

$$\mathcal{L}_{f,n} = \mathcal{L}_{g,n} \quad \forall n$$

f can be chosen so that $\exists\, C, \overline{C}$ (w/ reversed coding) s.t.

$|C|_f = \mu \neq |\overline{C}|_f = \lambda$. Then $|C|_g = \lambda$, $|\overline{C}|_g = \mu$.

Unmarked L.S.R.: positive results

Question: Are expanding circle maps locally rigid w.r.t. their length spectra?

Conjecture: Let $g: \mathbb{S}^1 \to \mathbb{S}^1$ be a C^r-smooth, $r > 1$, expanding circle map of degree $d \geq 2$. Then $\exists\ \varepsilon = \varepsilon(g)$ s.t.

If f is another such map w/

$$\|f-g\|_{C^r} \leq \varepsilon, \quad \mathcal{L}_{f,n} = \mathcal{L}_{g,n} \quad \forall n \in \mathbb{N},$$

then f and g are C^r-conjugate.

Unmarked L.S.R.: positive results

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If f is another such map so that

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then f and g are C^r-conjugate.

We establish this conjecture w/ some additional assumptions

Length spectral rigidity: main result

The length spectrum $\mathcal{L}_f$ is:

- β-sparse if $\exists\, \beta > 0$, $A > 0$ s.t.

$$\big|\, |C|_f - |C'|_f \,\big| \geq A\, d^{-\beta n} \quad \forall \text{ cycles } C \neq C'$$

$$\text{per}(C) = \text{per}(C') = n$$

[McMullen] β-sparse maps exist

Similar condition was studied for hyperbolic metrics by Dolgopyat-Jakobson.

<u>Thm [D-K]</u> $\exists\, \beta_0 > 0$ s.t. ~~the following holds~~

every $C^{r,1}$-smooth, $r \geq 1$, degree $d \geq 2$ expanding circle map with β-sparse length spectrum, $\beta < \beta_0$, is locally spectrally rigid.

Ideas of proof

- We want to conclude that $ML_f = ML_g$ if f is sufficiently close to g.
- Cannot do it in one step:

$$\boxed{ML_{f,n} : C \mapsto |C|_f \,, \quad per(C) \leq n}$$

marked length spectrum up to period n

Lemma: f, g

Recovering $\|f-g\|_{C^{1,1}} \leq \delta_2 < 1$, $\|f-g\|_{C^1} \leq \delta_1 < \delta_2$

$L_{f,n} = L_{g,n} \quad \forall n \in \mathbb{N}$

Length spectrum is β sparse

$$\Rightarrow \quad ML_{f,N} = M_{g,N} \quad \text{for} \quad N = \left\lfloor -\frac{\log \delta_1}{\beta+1} \right\rfloor$$

<u>Iterative recovering</u>: Fix g (reference map)

Construct a sequence $\{h_k\}_{k_0}$ of $C^{r,1}$-smooth coordinate adjustments s.t. for

$$f[k] = h_k \circ f \circ h_k^{-1}$$

$$\| f[k] - g \|_{C^{r,1}} \leq \varepsilon_0 , \quad \| h_k - id \|_{C^{r,1}} \leq \varepsilon_0$$

$$\| f[k] - g \|_{C^1} \leq \varepsilon[k] , \quad \| h_k - id \|_{C^1} \leq \varepsilon[k]$$

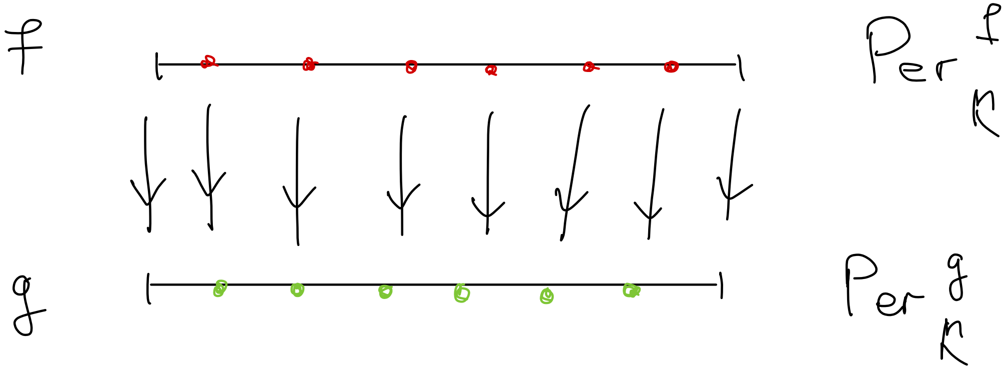
$$\sum_{k_0}^{\infty} \varepsilon[k] < \infty$$

Then $\psi = h_{k_0}^{-1} \circ h_{k_0+1}^{-1} \circ \dots$ is a smooth conjugacy

$\varepsilon_0 = \varepsilon_0(g)$ is chosen so that this scheme starts to work

How to construct h_K? (idea)

(1)



Extend the discrete map

$$h_K : Per^f_n \longrightarrow Per^g_n$$

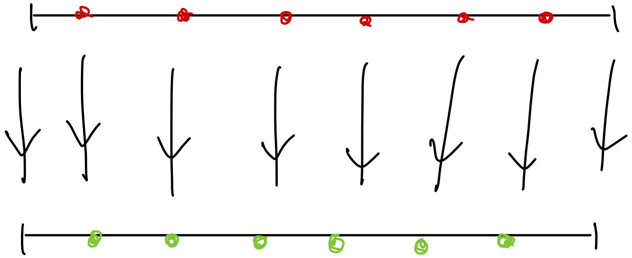
in $C^{r,1}$ fasion using

Whitney extension (check divided differences)

Why do we need this? If $Per^f_K = Per^g_K$, then $f-g$ vanishes on exponentially dense set of points, then $\|f-g\|_{C^1} \lesssim A \cdot \Lambda^{-Kr} \|f-g\|_{C^{r,1}}$, $\Lambda > 1$

→ can recover marking using sparsity

How to construct h_K? (idea)

(1) f  g

Per^f_n

Per^g_n

Extend the discrete map

$$h_K : \mathrm{Per}^f_n \longrightarrow \mathrm{Per}^g_n$$

in $C^{r,1}$ fasion using Whitney extension (check divided differences)

Why do we need this? If $\mathrm{Per}^f_K = \mathrm{Per}^g_K$, then $f-g$ vanishes on exponentially dense set of points, then $\|f-g\|_{C^1} \leq A \cdot \Lambda^{-Kr} \|f-g\|_{C^{r,1}}$, $\Lambda > 1$

→ can recover marking using sparsity

(2) Control on distortion of intervals? ⇝ Finite Livsic theorem

(quantitative solution to co-homological equation S. Katok '90, Gouëzel-Lefeuvre '02 ...

Thm [D-K.]: Assume f, g preserve Leb. measure on $\mathbb{S}^1$, and let ψ be the α-Hölder conjugacy between f and g. Then $\exists\ \Lambda > 1$ with the following property: if $\mathcal{ML}_{f,n} = \mathcal{ML}_{g,n}$, then

$$|f'(x) - g'(\psi(x))| \leq \Lambda^{-n} \|f-g\|_{C^0}^{1-\alpha}$$

The End