Generic dynamics on Oka-Stein manifolds and Stein manifolds with the density property

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Introduction

Joint work with Leandro Arosio, Università di Roma Tor Vergata

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Plan of the talk

1. The two dichotomies
2. Previous work that we build on
3. Our main results
4. Some sketches of proofs
5. Open questions

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Fatou set F_f: points with a nbhd U such that every subsequence of the sequence of iterates of f has a subsequence that converges locally uniformly on U to a holomorphic map into X or to ∞_X

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$\mathrm{rne}(f) \subset F_f$: open set of points $p \in X$ for which there are nbhds U of p in X and V of f in $\mathrm{End}\, X$ or $\mathrm{Aut}\, X$ and a compact subset K of X such that $g^j(U) \subset K$ for all $g \in V$ and $j \geq 0$

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Julia set: $J_f = X \setminus F_f$ for an endomorphism
$J_f = J_f^+ \cap J_f^- = X \setminus (F_f^+ \cup F_f^-)$ for an automorphism

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Equivalence relation on chain-recurrent set C_f: $p \sim q$ if for all ϵ, there is an ϵ-chain from p to q and an ϵ-chain from q to p

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Chain-recurrence is the weakest reasonable notion of recurrence

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Attractor determined by U: $A = \bigcap_{n \geq 0} \overline{f^n(U)}$ (closed)
Basin of A relative to U: $B(A, U) = \bigcup_{n \geq 0} f^{-n}(U) \supset A$ (open)
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We want to relate the Conley and Fatou-Julia decompositions

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• The weak closing lemma holds for endomorphisms of an Oka-Stein manifold

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The tameness requirement for automorphisms was an obstacle that we partly overcame by exploiting stable and unstable manifolds

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Let p be a saddle fixed point of an automorphism f of a Stein manifold X. The stable manifold of f through p is

$$W_f^s(p) = \{x \in X : f^j(x) \to p \text{ as } j \to \infty\}$$

$W_f^s(p) \subset J_f^+$ is an immersed submanifold, biholomorphic to $\mathbb{C}^k$
p and $W_f^s(p)$ vary continuously with f

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$F_f = \mathrm{rne}(f)$	$F_f^+ = \mathrm{rne}(f)$
$\Omega_f = J_f \cup \mathrm{att}(f)$	$\Omega_f = J_f \cup \mathrm{att}(f) \cup \mathrm{rep}(f)$
$J_f = \overline{\mathrm{sad}(f)}$ ($\dim X \geq 2$)	$J_f^* = \overline{\mathrm{sad}(f)} \subset J_f$ is the closure of the set of transverse homoclinic points
f is chaotic on J_f	J_f^* is the largest forward invariant subset of X on which f is chaotic

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Open question for generic autos (to do with tameness): Is $J_f^* = J_f$?

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For a generic C^1 diffeomorphism f of a compact smooth manifold, $C_f = \Omega_f$ (Bonatti-Crovisier 2004). Relies on a *connecting lemma*

Additional results

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1-dimensional result, using Baker (1975), but not the classification of Fatou components of transcendental entire functions:

For generic $f \in \operatorname{End}\mathbb{C}$, a point in F_f is attracted to an attracting cycle or lies in a dynamically bounded wandering domain

Every Fatou component of f is a disc

Sample proof sketch

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Hence

$$J_f = X \setminus \big(\operatorname{rne}(f) \cup \operatorname{rne}(f^{-1})\big) = \overline{W_f^s(q)} \cap \overline{W_f^u(q)}$$

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Conversely, let $x \in \Omega_f \setminus J_f$. By key technical result, $x \in \operatorname{rne}(f)$ or $x \in \operatorname{rne}(f^{-1})$. If $x \in \operatorname{rne}(f)$, then $x \in \operatorname{att}(f)$ by closing lemma. Similarly, if $x \in \operatorname{rne}(f^{-1})$, then $x \in \operatorname{att}(f^{-1}) = \operatorname{rep}(f)$

Open questions

For endomorphisms Are all Fatou components of a generic endomorphism f pre-recurrent? Equivalently (not trivially), is the Fatou set of f the union of the basins of attraction?

Weak closing lemma for chain-recurrent points $\Rightarrow$ yes

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However, we have proved that endomorphisms with a non-pre-recurrent Fatou component are dense in End X!

For automorphisms If a certain strong closing lemma holds and if a generic auto has no non-recurrent Fatou components, then for a generic auto f, $J_f = J_f^*$ is the complement of the union of the basins of attraction of the attracting and repelling cycles of f