Holomorphic factorization of vector bundle automorphisms

Frank Kutzschebauch, Universität Bern

Portoroz — June 2024

Outline

Any matrix $A \in SL_n(\mathbb{C})$ is a product of elementary matrices of the form

$$Id + a_{ij}E_{ij} = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & a_{ij} & 1 & & 0 \\ \vdots & & \vdots & \ddots & \\ 0 & & 0 & & 1 \end{pmatrix}$$

or equivalently a product of upper and lower triangular unipotent matrices.

$$A = \begin{pmatrix} 1 & 0 \\ G_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2 \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_N \\ 0 & 1 \end{pmatrix}, \textit{where} \quad G_i \in \mathbb{C}^{n(n-1)/2}$$

Proof: Gauss elimination, it requires:
1.) Adding multiples of a row to another row
2.) Interchange of rows :

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

3.) multiplication of rows by constants:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a^{-1}-1 \\ 0 & 1 \end{pmatrix}$$

(Whitehead lemma)

What if the matrix A depends on a parameter x (continuously, polynomially, holomorphically)? Can the upper and lower triangular unipotent matrices be chosen depending well on the parameter?

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & G_N(x) \\ 0 & 1 \end{pmatrix}$$

Now the G_i are maps $G_i : X \to \mathbb{C}^{n(n-1)/2}$.

- Let $R = \{f : X \to \mathbb{C}\}$ denote the ring of continuous/ polynomial /holomorphic functions on a topological space/ algebraic variety / complex space X.
- In the language of K-theory we are asking about factorization of $SL_n(R)$ (special linear group over the ring R) as product of elementary matrices over that ring.
- Given $m \geq 2$ and an associative, commutative, unital ring R, let $E_n(R)$ denote the set of those $n \times n$ matrices which are representable as products of unipotent matrices with entries in R. We ask about the relation of $E_n(R)$ and $SL_n(R)$.
- The obstruction to this factorization is called the special K_1-group of the ring R, (more precise the n-th, where n is the size of the matrices).

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53

- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53

- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Algebraic results

- $SL_n(\mathbb{C}[z_1])$ factorizes, more generally for Euclidean rings R $SL_n(R)$ factorizes, however there is no universal bound on the number of factors! van der Kallen, W., $SL_3(\mathbb{C}[X])$ does not have bounded word length, Algebraic K-theory, Part I (Oberwolfach, 1980), Lecture Notes in Math., 966, 357–361, 1982
- $SL_2(\mathbb{C}[z_1, z_2, \ldots, z_n])$ does not factorize for $n \geq 2$ counterexample found by Cohn (1966)

$$\begin{pmatrix} 1 - z_1 z_2 & z_1^2 \\ -z_2^2 & 1 + z_1 z_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}[z_1, z_2])$$

P. M. Cohn, On the structure of the GL_2 of a ring, *Inst. Hautes Études Sci. Publ. Math.* (1966), no. 30, 5–53

- $SL_n(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 3$

A. A. Suslin, The structure of the special linear group over rings of polynomials, *Izv. Akad. Nauk SSSR Ser. Mat.* 41 (1977), no. 2, 235–252, 477

Symplectic results

- $Sp_{2n}(\mathbb{Z}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 2$

  Grunewald, Fritz; Mennicke, Jens; Vaserstein, Leonid On symplectic groups over polynomial rings. Math. Z. 206 (1991)

- $Sp_{2n}(\mathbb{C}[z_1, z_2, \ldots, z_m])$ does factorize for all m and all $n \geq 2$

  Kopeiko, V. I. On the structure of the symplectic group of polynomial rings over regular rings. (Russian) Fundam. Prikl. Mat. 1 (1995), no. 2, 545–548

  Kopeiko, V. I. Stabilization of symplectic groups over a ring of polynomials. (Russian) Mat. Sb. (N.S.) 106(148) (1978), no. 1, 94–107

- $Sp^0_{2n}(\mathcal{O}(X))$ factorizes on a Stein space X
  Ivarsson, B.; Kutzschebauch, F.; Løw, Erik Holomorphic factorization of mappings into $Sp_{\mathbf{4}}(\mathbb{C})$, Anal. PDE, 16, 2023, 1, 233–277
  Schott, J. Holomorphic factorization of mappings into $Sp_{\mathbf{2}n}(\mathbb{C})$, arXiv:2207.05389, to appear in J. Eur. Math. Soc.

Topological results

- $SL_n(Cont(\mathbb{R}^3))$ factorizes

  W. Thurston and L. Vaserstein, On K_1-theory of the Euclidean space, *Topology Appl.* **23** (1986), no. 2, 145–148

- A general observation:

$$A_t(x) = \begin{pmatrix} 1 & 0 \\ tG_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & tG_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & tG_N(x) \\ 0 & 1 \end{pmatrix} \; t \in [0,1]$$

gives a homotopy of the map $A : X \to SL_m(\mathbb{C})$ to a constant map. Such maps are called null-homotopic. **If a map factorizes, then it is necessarily null-homotopic.**

Topological results

- $SL_n(Cont(\mathbb{R}^3))$ factorizes

  W. Thurston and L. Vaserstein, On K_1-theory of the Euclidean space, *Topology Appl.* **23** (1986), no. 2, 145–148

- A general observation:

$$A_t(x) = \begin{pmatrix} 1 & 0 \\ tG_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & tG_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & tG_N(x) \\ 0 & 1 \end{pmatrix} \; t \in [0,1]$$

gives a homotopy of the map $A : X \to SL_m(\mathbb{C})$ to a constant map. Such maps are called null-homotopic. **If a map factorizes, then it is necessarily null-homotopic.**

Continuous results

Theorem (Vaserstein)

For any natural number n and an integer $d \geq 0$ there is a natural number K such that for any finite dimensional normal topological space X of dimension d and null-homotopic continuous mapping $A\colon X \to SL_n(\mathbb{C})$ the mapping can be written as a finite product of no more than $K = K(d, n)$ unipotent matrices. That is, one can find continuous mappings $G_l\colon X \to \mathbb{C}^{n(n-1)/2}$, $1 \leq l \leq K$ such that

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \dots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

L. Vaserstein, Reduction of a matrix depending on parameters to a diagonal form by addition operations, *Proc. Amer. Math. Soc.* **103** (1988), no. 3, 741–746

Theorem (Hultgren, Wold)

Let X be a locally finite finite dimensional CW-complex and let $\pi : V \to X$ be a real (resp. complex) topological vector bundle of rank n. Assume that $n \geq 3$ (resp.$n \geq 2$) and let S be a (continuous) nullhomotopic special vector bundle automorphism of V . Then there exist unipotent vector bundle automorphisms $E_1, ..., E_N$ such that $S = E_N \circ \cdots \circ E_1$.

J. Hultgren, E.F. Wold, Unipotent Factorization of Vector Bundle Automorphisms. Int. J. of Math. Vol. 32, No. 03, 2150013 (2021)

The Gromov-Vaserstein problem for SL_n

Theorem

Let X be a finite dimensional reduced Stein space and $A\colon X \to SL_n(\mathbb{C})$ be a holomorphic mapping that is null-homotopic. Then there exist a natural number $K = K(dimX, n)$ and holomorphic mappings $G_1, \ldots, G_K\colon X \to \mathbb{C}^{n(n-1)/2}$ such that A can be written as a product of upper and lower diagonal unipotent matrices

$$A(x) = \begin{pmatrix} 1 & 0 \\ G_1(x) & 1 \end{pmatrix} \begin{pmatrix} 1 & G_2(x) \\ 0 & 1 \end{pmatrix} \ldots \begin{pmatrix} 1 & G_K(x) \\ 0 & 1 \end{pmatrix}$$

for every $x \in X$.

Ivarsson, B., Kutzschebauch, F. *Holomorphic factorization of mappings into $SL_n(\mathbb{C})$*, Ann. of Math. (2) 175 (2012), no. 1, 45-69

The holomorphic vector bundle case

Theorem (Ionita, K.)

Let X be a Stein space and $E \to X$ a rank 2 holomorphic vector bundle over X. Then $F \in \mathrm{SAut}(E)$ is a (finite) product of unipotent holomorphic automorphisms $u_i \in \mathrm{U}(E)$, $i = 1, 2, \ldots, K$,

$$F(x) = u_1(x) \cdot u_2(x) \cdot \ldots \cdot u_K(x)$$

if and only if F is null-homotopic.

Ionita, G.; Kutzschebauch, F. Holomorphic Factorization of vector bundle automorphisms. arXiv:2305.04350 2023

construction of the unipotent automorphisms

- $s_1, s_2 \in \Gamma_{\text{hol}}(E, X)$ linear independent outside a proper analytic subset A of X
- trivialization of E, $X = \cup_{i=1}^{\infty} U_i$ with a cocycle of transition functions $f_{i,j} : U_{i,j} \to \mathrm{GL}_2(\mathbb{C})$
- $s_1^i, s_2^i : U_i \to \mathbb{C}^2$ - local representations of the sections
- $A \cap U_i = \{x \in U_i : \det(s_1^i(x), s_2^i(x)) = 0\}$
- $\alpha_{i,j} := \det(f_{i,j}) = \det(s_1^j(x), s_2^j(x))(\det(s_1^i(x), s_2^i(x)))^{-1} : U_{i,j} \to \mathbb{C}^*$ defines a line bundle L
- any global holomorphic section in L^{-1} gives a holomorphic function $f \in \mathcal{O}(X)$ with the property that on U_i, the quotient $f(x)(\det(s_1^i(x), s_2^i(x)))^{-1}$ is holomorphic

define a global nilpotent holomorphic endomorphism N^- by

$$N^-(s_1(x)) = f(x)s_2(x) \text{ and } N^-(s_2(x)) = 0$$

in points x where $s_1(x), s_2(x)$ form a basis of E_x ,i.e., outside A.
It extends to the points in A, since

$$s_1^i(x) = \begin{pmatrix} a(x) \\ c(x) \end{pmatrix} \text{ and } s_2^i(x) = \begin{pmatrix} b(x) \\ d(x) \end{pmatrix},$$

then in the standard basis $e_1, e_2 \in \mathbb{C}^2$ the matrix of N^- is given by

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ f(x) & 0 \end{pmatrix} \cdot \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}^{-1} =$$

$$= \frac{f(x)}{\det(s_1^i(x), s_2^i(x))} \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} d(x) & -b(x) \\ -c(x) & a(x) \end{pmatrix}.$$

Proposition

Let $\pi : E \to X$ be a holomorphic vector bundle of rank 2 over a Stein space X of dimension n. Then there exist $n+1$ pairs of nilpotent holomorphic automorphism N_i^+, N_i^- in $\mathrm{End}(E)$, $i = 1, 2, \ldots, n+1$ together with (non-zero) holomorphic functions $f_i \in \mathcal{O}(X)$, without common zeros, with the property that on each of the sets $X \setminus \{f_i = 0\}$ the bundle E is trivial and locally the pair (N_i^+, N_i^-) is holomorphically conjugated on $X \setminus \{f_i = 0\}$ to the "standard" pair

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Theorem

Let X be a Stein space and $f_i \in \mathcal{O}(X)$, $i = 1, 2, \ldots, m$ finitely many holomorphic functions without common zeros. Then every null-homotopic special holomorphic vectorbundle automorphism $F \in \mathrm{SAut}(E)$ can be written as a product

$$F = \prod_{i=1}^{m} G_i,$$

where each of the $G_i \in \mathrm{SAut}(E)$ is a special holomorphic vectorbundle automorphism, whose difference to the identity is divisible by f_i^4. Moreover, the G_i's are strongly null-homotopic, i.e., they are null-homotopic such that on $\{f_i = 0\}$ the homotopy $(G_i)_t = \mathrm{Id}$, $\forall t \in [0, 1]$.

Denote $A_i := \{f_i = 0\}$ for any U open in X define

$$\mathcal{F}_i(U) := \{\alpha \in \mathrm{SAut}_{\mathsf{hol}}(E|_U) : f_i \mid (\alpha - \mathsf{Id})\},$$

and similarily sheaves $\mathcal{G}_i$ by

$$\mathcal{G}_i(U) := \{\alpha \in \mathrm{SAut}_{\mathsf{cont}}(E|_U) : \alpha|_{U \cap A_i} = \mathsf{Id})\}.$$

$$\Phi(U) := \{(\alpha_1, \alpha_2, \dots \alpha_m) \in \mathcal{F}_1(U) \times \dots \times \mathcal{F}_m(U) : \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_m = f|_U\}$$

and similarily the sheaf Ψ as

$$\Psi(U) := \{(\alpha_1, \alpha_2, \dots \alpha_m) \in \mathcal{G}_1(U) \times \dots \times \mathcal{G}_m(U) : \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_m = f|_U\}.$$

The (fibrewise) map $\mathrm{SAut}(E)^{m-1} \times \mathrm{SAut}(E)^m \to \mathrm{SAut}(E)^m$ given by

$$(\beta_1(x), \beta_2(x), \ldots, \beta_{m-1}(x)) \times (\alpha_1(x), \alpha_2(x), \ldots, \alpha_m(x)) \mapsto \\ (\alpha_1(x)\beta_1(x)^{-1}, \beta_1(x)\alpha_2(x)\beta_2(x)^{-1}, \ldots, \beta_{m-1}(x)\alpha_m(x))$$

is a "thickening" for the sub sheaf $\mathcal{H}_f$ of holomorphic sections of $\mathrm{SAut}(E)^m$ defined by

$$\mathcal{H}_f(U) := \{(\alpha_1, \alpha_2, \ldots, \alpha_m) : \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_m = f|_U\}.$$

The subsheaf $\mathcal{J}$ of $\mathrm{SAut}(E)^{m-1}$ given by

$$\mathcal{J} = \{(\beta_1, \beta_2, \ldots, \beta^{m-1}) \in \mathrm{SAut}(E)^{m-1} :$$
$$f_1 \mid (\beta_1 - \mathsf{Id}), f_2 \mid (\beta_1^{-1}\beta_2 - \mathsf{Id}), \ldots, f_{m-1} \mid (\beta_{m-2}^{-1}\beta_{m-1} - \mathsf{Id}), f_m \mid (\beta_{m-1} - \mathsf{Id})$$

leaves the sheaf Φ invariant.
Define the subsheaf $\mathsf{Lie}(\mathcal{J})$ of $\mathrm{End}^0(E)^{m-1}$ corresponding to $\mathcal{J}$ as

$$\mathsf{Lie}(\mathcal{J}) = \{(v_1, v_2, \ldots, v_{m-1}) \in \mathrm{End}^0(E)^{m-1} :$$
$$f_1 \mid v_1, f_2 \mid (v_1 - v_2), \ldots, f_{m-1} \mid (v_{m-2} - v_{m-1}), f_m \mid v_{m-1}\}, \quad (1)$$

which is a coherent subsheaf of $\mathrm{End}^0(E)^{m-1}$.

The map

$$v = (v_1, v_2, \ldots, v_{k-1}) \mapsto a(x)\exp(v) \\ := (\alpha_1(x)\exp(-v_1), \exp(v_1)\alpha_2(x)\exp(-v_2), \ldots, \\ \exp(v_{l-1})\alpha_l(x)\exp(-v_l), \ldots, \exp(v_{m-1})\alpha_k(x)$$

is the required (even global) spray around the section $a = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Phi$. The transition function now comes from $\log(a(x)\exp(v(x)))$ and the gluing can be achieved as in the paper of Studer. Alternatively, by the method of Cartan-Grauert as explained by Forster and Ramspott.

O. Forster, K. J. Ramspott, *Okasche Paare von Garben nicht-abelscher Gruppen.* Invent. Math. 1, 260-286 (1966), Studer, Luca, *A splitting lemma for coherent sheaves.* Anal. PDE 14 (2021), no. 6, 1761Ð-1772.

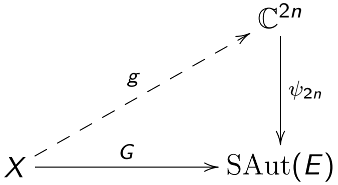
define $U^{\pm}(h) = \mathrm{Id} + h \cdot N^{\pm}$ for a function h on X, where $U^{\pm} = \mathrm{Id} + N^{\pm}$

Theorem

Let U^{+}, U^{-} and $f \in \mathcal{O}(X)$ be as above and let $G \in SAut(E)$ with the properties that $f^4 \mid G - \mathrm{Id}$ and G is strongly null-homotopic. Then

$$G = U^{-}(h_1) \cdot U^{+}(h_2) \cdot \ldots \cdot U^{-}(h_{2k-1}) \cdot U^{+}(h_{2k})$$

for some integer k and holomorphic functions $h_i \in \mathcal{O}(X)$, $i = 1, 2, \ldots, k$.



where $\psi_{2n} : \mathbb{C}^{2n} \to \mathrm{SAut}(E)$ is given by

$$(z_1, \ldots, z_{2n}) \longmapsto U^-(z_1) \cdot U^+(z_2) \cdot \ldots U^-(z_{2n-1}) \cdot U^+(z_{2n}).$$

Proposition

Let $f \in \mathcal{O}(X)$ be a holomorphic function on a complex space X and let $n \geq 2$. Given a holomorphic map $G : X \to \mathrm{SAut}(E)$ with the property that $G - \mathrm{Id}$ is divisible by f^3, then the reachable points of the fibration $(G^*(\mathbb{C}^{2n}) \setminus \mathrm{Sing}, G^*\psi_{2n}, X)$ form a stratified elliptic submersion.

$$G(x) = U^-(z_1(x)) \cdot U^+(z_2(x)) \cdot \ldots \cdot U^+(z_{2n}(x)),$$

in coordinates:

$$S_i(x)^{-1}\,\mathrm{Mat}(G)(x)S_i(x) = S_i(x)^{-1}U^-(z_1(x))S_i(x) \cdot S_i^{-1}(x)U^+(z_2(x))S_i(x$$

$$\mathrm{Id} + f^3(x)S_i^{-1}(x)\begin{pmatrix} a_1(x) & b_1(x) \\ c_1(x) & d_1(x) \end{pmatrix} S_i(x) = \begin{pmatrix} 1 & 0 \\ z_1 f(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z_2 f(x) \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Id} + f^2(x) f_i(x) S_i^{\#}(x) \begin{pmatrix} a_1(x) & b_1(x) \\ c_1(x) & d_1(x) \end{pmatrix} S_i(x) =$$

$$= \begin{pmatrix} 1 & 0 \\ z_1 f(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z_2 f(x) \\ 0 & 1 \end{pmatrix} \cdot \ldots \cdot$$

$$\mathrm{Id} + f^2(x)\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1 f(x) & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & z_2 f(x) \\ 0 & 1 \end{pmatrix} \cdot \ldots \cdot$$

$$\begin{pmatrix} Q_{11}^k(z,f) & Q_{12}^k(z,f) \\ Q_{21}^k(z,f) & Q_{22}^k(z,f) \end{pmatrix} := \begin{pmatrix} 1 & z_2 f \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ z_3 f & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 1 & 0 \\ z_{2k+1} f & 1 \end{pmatrix}$$

Then

$$Q_{11}^k(z,f) = 1 + f^2 \sum_{1 \le i \le j \le k} z_{2i} z_{2j+1} + f^3 \widetilde{Q}_{11}^k(z,f).$$

$$Q_{12}^k(z,f) = f \sum_{1 \le i \le k} z_{2i} + f^3 \widetilde{Q}_{12}^k(z,f),$$

$$Q_{21}^k(z,f) = f \sum_{1 \le j \le k} z_{2j+1} + f^3 \widetilde{Q}_{21}^k(z,f),$$

$$Q_{22}^k(z,f) = 1 + f^2 \sum_{1 \le j < i \le k} z_{2i} z_{2j+1} + f^3 \tilde{Q}_{22}^k(z,f).$$

Theorem

Let U^+, U^- and $f \in \mathcal{O}(X)$ be as above and consider a continuous bundle automorphism $G \in \mathrm{SAut}_{\mathrm{top}}(E)$ with the properties that $f^4 \mid G - \mathrm{Id}$ and G is strongly null-homotopic. Then

$$G = U^-(h_1) \cdot U^+(h_2) \cdot \ldots \cdot U^-(h_{2k-1}) \cdot U^+(h_{2k})$$

for some integer k and continuous functions $h_i \in \mathcal{O}(X)$, $i = 1, 2, \ldots, k$.

Novelties (1)

For

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \in \mathrm{SL}_2(\mathcal{C}^{\mathbb{C}}(X)),$$

under the special condition $a(x) \neq 0$ one can solve the equation

$$\begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_1(x) & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 1 & z_4(x) \\ 0 & 1 \end{pmatrix}$$

with the interpolation condition

$$A(x) = \mathrm{Id} \implies z_1(x) = \cdots = z_4(x) = 0$$

Solution:

- $z_1(x) = \dfrac{c(x) - \dfrac{a(x)-1}{\sqrt{|a(x)-1|}}}{a(x)};$
- $z_2(x) = \sqrt{|a(x)-1|};$
- $z_3(x) = \dfrac{a(x)-1}{\sqrt{|a(x)-1|}};$
- $z_4(x) = \dfrac{b(x) - \sqrt{|a(x)-1|}}{a(x)}.$

Novelties (2)

In case of divisibility $f^3 \mid (A - \mathrm{Id})$, one can solve the equation

$$\begin{pmatrix} 1+f^3a & f^3b \\ f^3c & 1+f^3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ fz_1 & 1 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} 1 & fz_4 \\ 0 & 1 \end{pmatrix}$$

under the assumption that $1 + f^3a \neq 0$ with the interpolation condition

$$A(x) = \mathrm{Id} \implies z_1(x) = \cdots = z_4(x) = 0$$

Solution

- $z_1 = \dfrac{f^2 c - \dfrac{af}{\sqrt{|af|}}}{1 + f^3 a}$;
- $z_2 = \sqrt{|af|}$;
- $z_3 = \dfrac{af}{\sqrt{|af|}}$;
- $z_4 = \dfrac{f^2 b - \sqrt{|af|}}{1 + f^3 a}$.

Novelties (3)

Special Whitehead Lemma

Writing a determinant 1 invertible diagonal matrix D as a product of four elementary matrices over the ring of complex valued continuous matrices with the interpolation condition

$$D(x) = \mathrm{Id} \implies z_1(x) = \cdots = z_4(x) = 0$$

Namely: For $\lambda \in \mathcal{C}^{\mathbb{C}}(X)^*$, the following holds:

$$D(x) = \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ \frac{1-\lambda}{\sqrt{|\lambda-1|}} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \sqrt{|\lambda-1|} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{\lambda-1}{\sqrt{|\lambda-1|}} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\frac{\sqrt{|\lambda-1|}}{\lambda} \\ 0 & 1 \end{pmatrix}$$

THANK YOU!