A characterization of rational convexity in Stein and projective manifolds

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Rational convexity in $\mathbb{C}^n$

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- Classical motivation.
  - ∗ Theorem A. Every holomorphic function on a neighborhood of K is uniformly approximable on K by rational functions with no poles on K.
  - ∗ If every continuous function on a neighborhood of K is uniformly approximable on K by rational functions with no poles on K, then K is rationally convex.

The pluripotential-theoretic point of view (Duval–Sibony)

- Yet another characterization. Given a compact set $K \subset \mathbb{C}^n$,

$$z \notin r(K) \quad \Longleftrightarrow \quad \text{there is a (weakly) positive closed current } T \text{ of bidegree } (1,1) \text{ such that } z \in \operatorname{supp} T \text{ but } \operatorname{supp} T \cap K = \emptyset.$$

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- **Theorem B.** Let T be a p.c.c. of bidegree $(1,1)$ on $\mathbb{C}^n$ s.t. $\mathbb{C}^n \setminus \operatorname{supp}\ T \Subset \mathbb{C}^n$.

$$K_s = \{z \in \mathbb{C}^n : \operatorname{dist}(z, \operatorname{supp}\ T) \geq s\}, \quad s > 0,$$

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Appl. Any p.c.c. T of bidegree $(1,1)$ can be weakly approx. by a sequence of "rational divisors", i.e., $[H_j]/N_j$, where H_j is a hypersurface in $\mathbb{C}^n$ and $N_j \in \mathbb{Z}$. Moreover, H_j's converge to supp T in the Hausdorff metric.

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- Theorem C. Suppose $j : S \hookrightarrow \mathbb{C}^n$ is a smooth totally real submanifold. Then,

$$r(S) = S \iff S \text{ is isotropic w.r.t some Kähler form } \omega \text{ on } \mathbb{C}^n\text{, i.e., } j^*\omega = 0.$$

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Gen. Immersions with special singularities (Gayet, Duval–Gayet, Shafikov–Sukhov, Mitrea).

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- Theorem D. Suppose $K \subset \mathbb{C}^n$ is a compact set such that

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Question. What are the analogues of these results in more general complex manifolds?

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- A hypersurface Y in a symplectic manifold (X, ω) is said to be of contact-type w.r.t. ω if there is a contact form λ on Y such that $d\lambda = \omega|_Y$.

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- A Weinstein domain (X, ω, V, f) consists of
  * a compact symplectic manifold (X, ω) with boundary,
  * a globally-defined Liouville v.f. V which points transverally out of ∂X.
  * a Morse function $f : X \longrightarrow \mathbb{R}$, locally constant on ∂X, s.t. V is gradient-like for f, i.e.,

$$V(f) \geq \delta(|V|^2 + |df|^2), \quad \text{for some } \delta > 0.$$

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Question. Which smooth, closed, oriented manifolds can be realized (up to diff.) as contact type hypersurfaces in $(\mathbb{R}^{2n}, \omega_{std})$?

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  - ∗ str ψcvx domains in $\mathbb{C}^2$
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  $\exists\, X \subset \mathbb{C}^2$ that isotopes to a str ψcvx, but not rat cvx domain in $\mathbb{C}^2$

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- Example:

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- Gompf's (open) conjecture (2013). No nontrivial Brieskorn integer homology 3-sphere bounds a str ψcvx domain in $\mathbb{C}^2$.

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Further remarks

A characterization of all rationally convex sets. Let X be a Stein manifold, and $K \subset X$ be a compact set. Then K is (strongly) meromorphically convex in X if and only if there exists a neighborhood basis of K such that every element of the basis is of the form $\Omega = \{\rho < 0\}$, where ρ is a strictly plurisubharmonic function in a neighborhood of $\overline{\Omega}$, and $dd^c\rho$ extends off a neighborhood of $\overline{\Omega}$ to a (trivial) Hodge form on X. The same characterization holds for rational convexity of proper compact subsets of a projective manifold.

Convexity w.r.t. currents. Meromorphic convexity in Stein manifolds is equivalent to convexity with respect to p.c.c. T of bidegree $(1,1)$ such that $[T] \in H^2(X, \mathbb{Z})$.

Two key ingredients

Lemma A. Let (X, ω) be a projective manifold endowed with a Kähler form. Let $K \subset X$ be rationally convex.

(i) For every $z \notin K$, $\exists$ p.c.c. T of bidegree $(1, 1)$ such that $[T] \in H^2(X, \mathbb{Z})$, T admits a continuous potential, is smooth and positive at z and vanishes in a neighborhood of K.

(ii) for every $\varepsilon > 0$ and rel cpt nbhd V of K, there exists a smooth closed $(1, 1)$-form ω_ε such that

- $\omega_\varepsilon \geq \omega$ on $X \setminus V$
- $\omega_\varepsilon \equiv 0$ on a neighborhood of K
- $\omega_\varepsilon \geq -\varepsilon\omega$ on V
- $[\omega_\varepsilon] \in H^2(X, \mathbb{Z})$.

Rmk. Theorem B from Duval–Sibony doesn't generalize to all projective X.

Lemma B. Let X be a projective manifold and $V \subset X$ be an open subset. Let L be a positive line bundle on X and φ a positive continuous metric of L on X. Let $s \in \mathcal{O}(V, L)$. Suppose

$$K = \{z \in V : \|s\|_\varphi \geq 1\}$$

is compact. Then, K is rationally convex.

THANK YOU.