WANDERING DOMAINS and BLASCHKE SEQUENCES via QC SURGERY

Complex Analysis, Geometry and Dynamics III

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HOLOMORPHIC DYNAMICS: THE BASIC PARTITION

We consider dynamics of $f : \mathbb{C} \to \mathbb{C}$ **entire transcendental.**

The complex plane decomposes into two **totally invariant sets**:

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  - The boundary between the different stable domains

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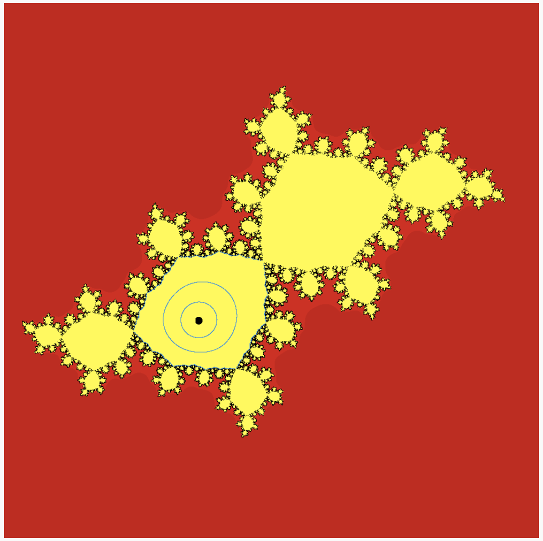

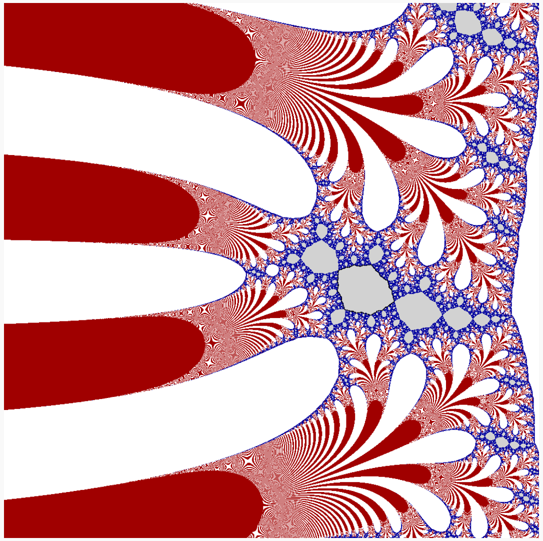

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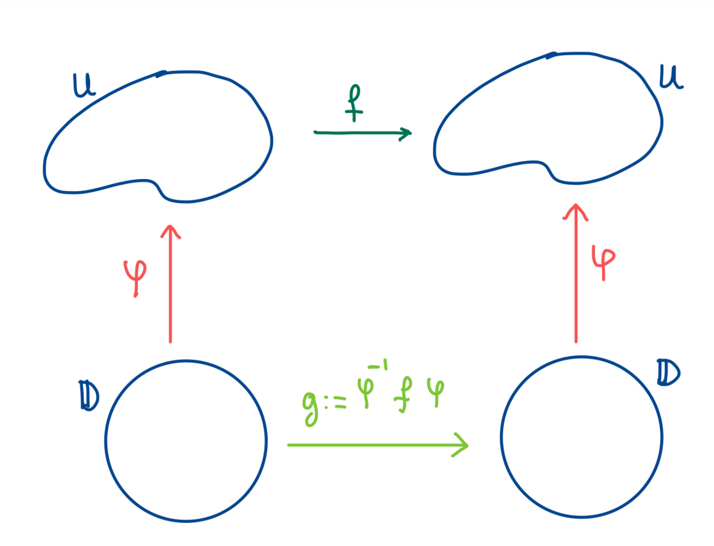
- Three possibilities:

$$\begin{cases} \textbf{Periodic} & \text{if } f^p(U) = U \text{ for some } p \geq 1 \\ \textbf{Preperiodic} & \text{if } f^k(U) \text{ is periodic for some } k > 1 \text{ (and not earlier)} \\ \textbf{Wandering} & \text{if } f^k(U) \text{ is not periodic for any } k \geq 1. \end{cases}$$

PERIODIC FATOU COMPONENTS and INNER FUNCTIONS

If U is a (simply connected) Fatou component and $\varphi : \mathbb{D} \to U$ is the Riemann map, then $g := \varphi^{-1} \circ f \circ \varphi$ is an **associated inner function** to f (holomorphic self-map of $\mathbb{D}$ with a.a. radial limits in $\partial\mathbb{D}$).



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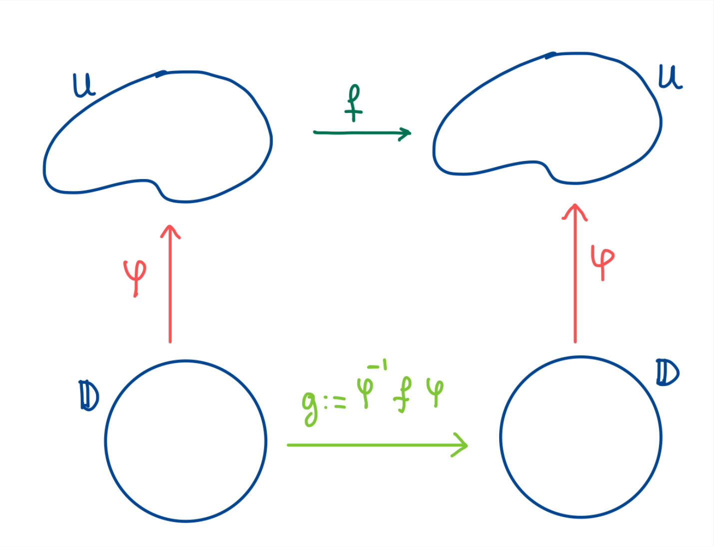

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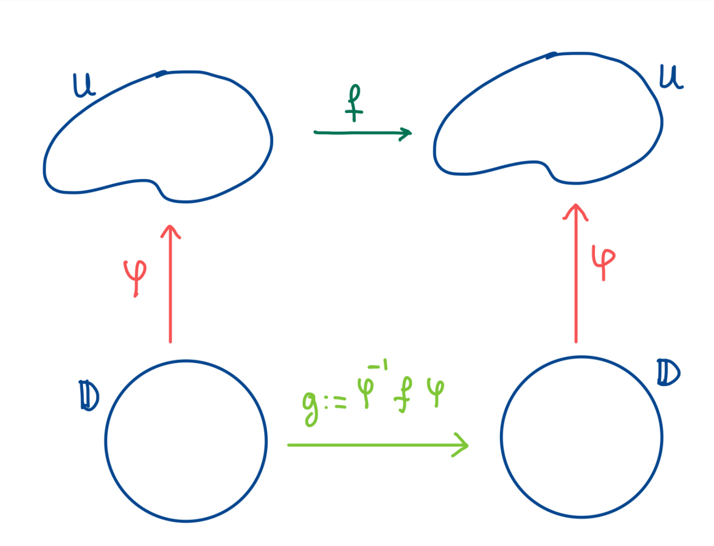

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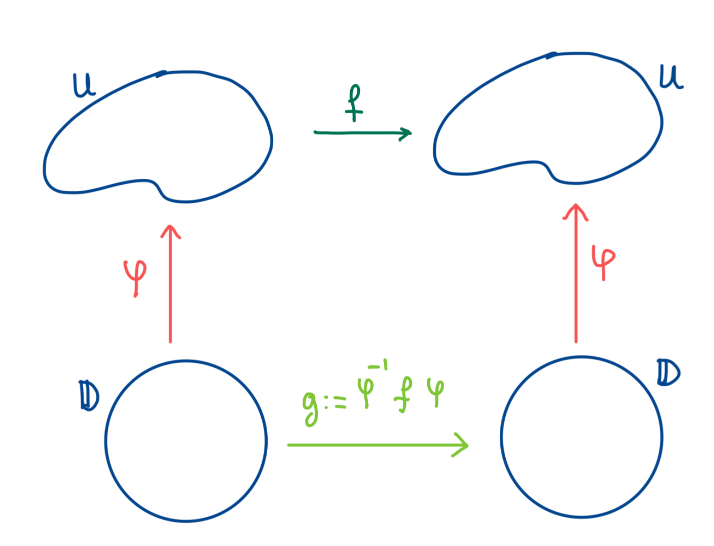

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- Transfering results from g to f is not always possible.
- If $\deg f < \infty$ then g is a finite Blasche product.

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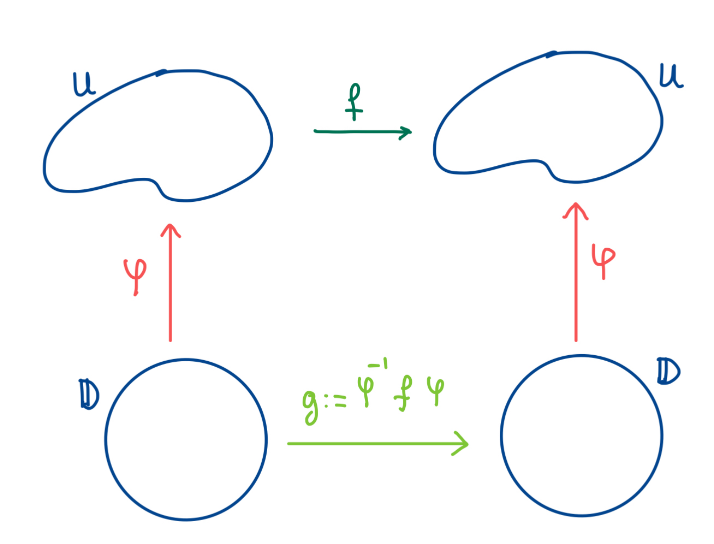

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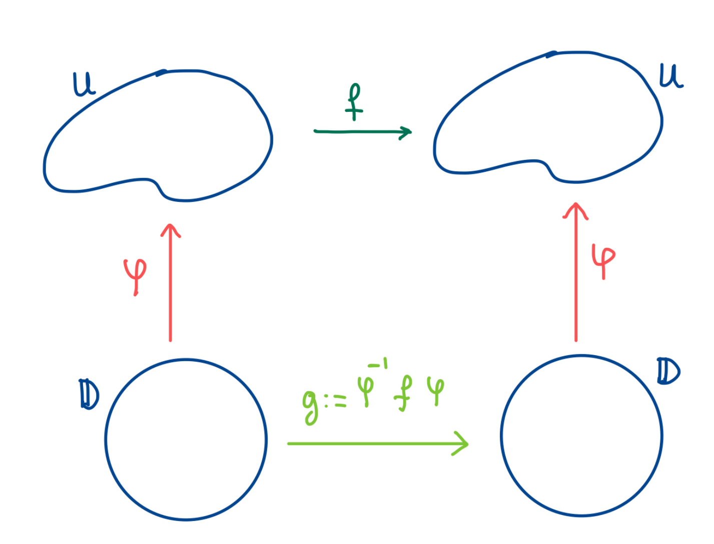

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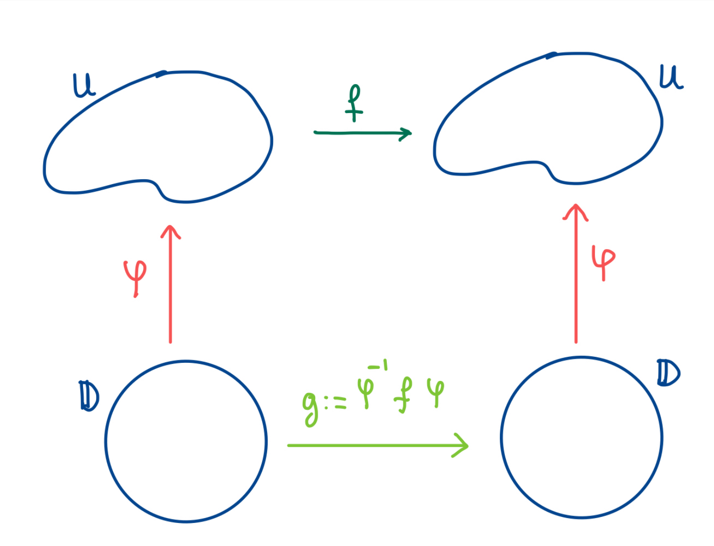

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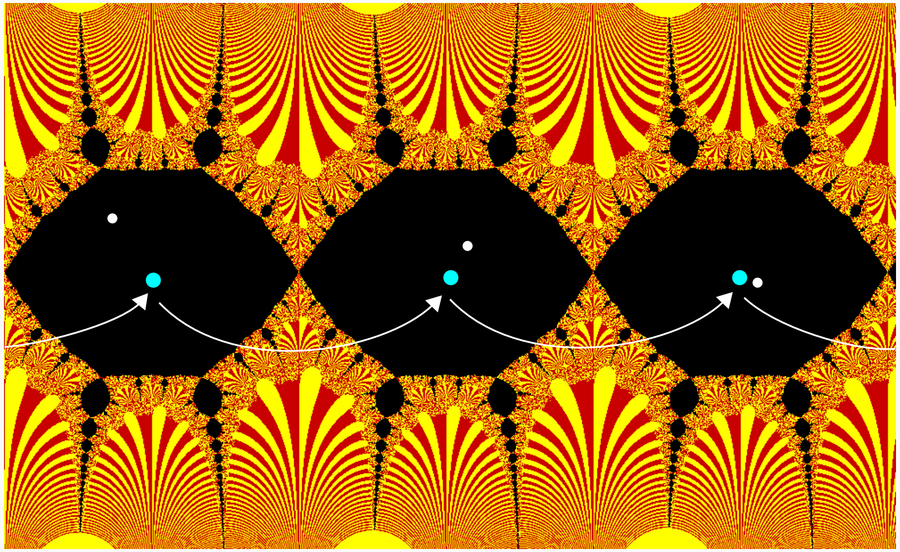
A more systematic study in [Evdoridou-Rempe-Sixsmith'20]

WANDERING DOMAINS

As for Baker domains, **wandering domains** (= **wandering components**) can only exist for **transcendental maps** (i.e. with essential singularities).

- U is a **wandering domain** if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.



$z + 2\pi + \sin(z)$

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- They do not exist for maps with a finite number of **singular values** (singularities of f^{-1}).
- **Major open questions** related to wandering domains.
- Need to understand dynamics **inside** the WD [Benini-Evdoridou-F-Rippon-Stallard'22], **on their boundary** [BEFRS'23,24] and also the relation of the WD with the **singular values** of the map (all related [Baranski-F-Jarque-Karpinska'17])

SIMPLY CONNECTED WANDERING DOMAINS

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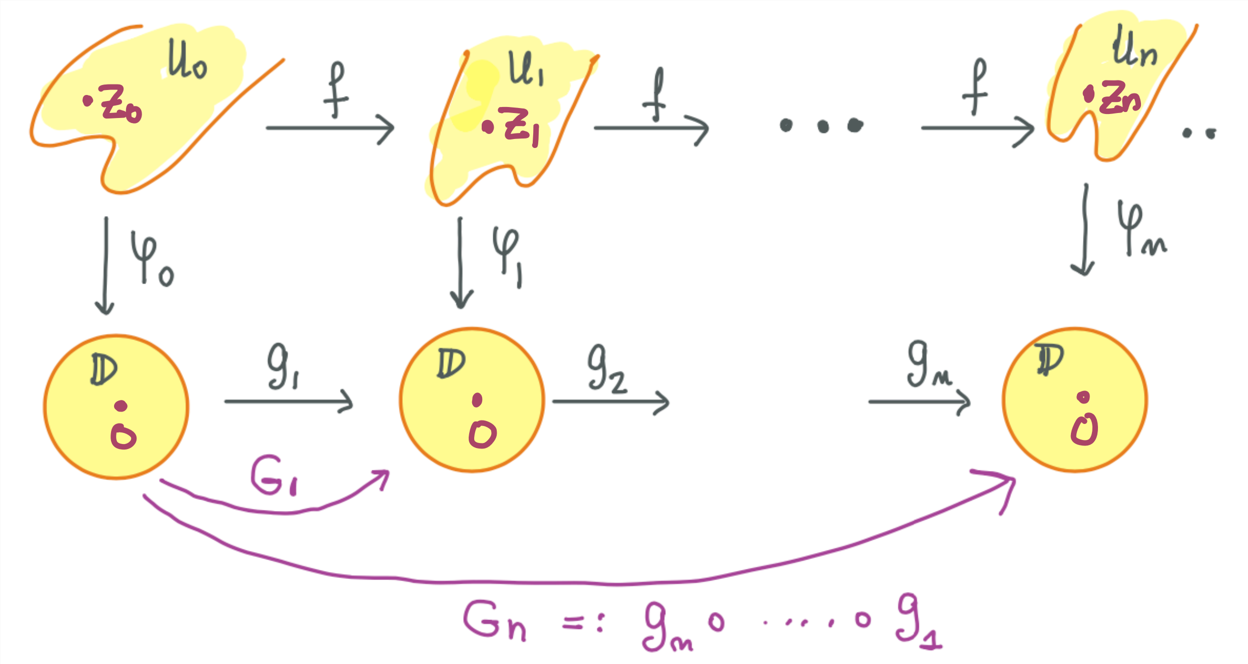
(Other choices are also possible!!)



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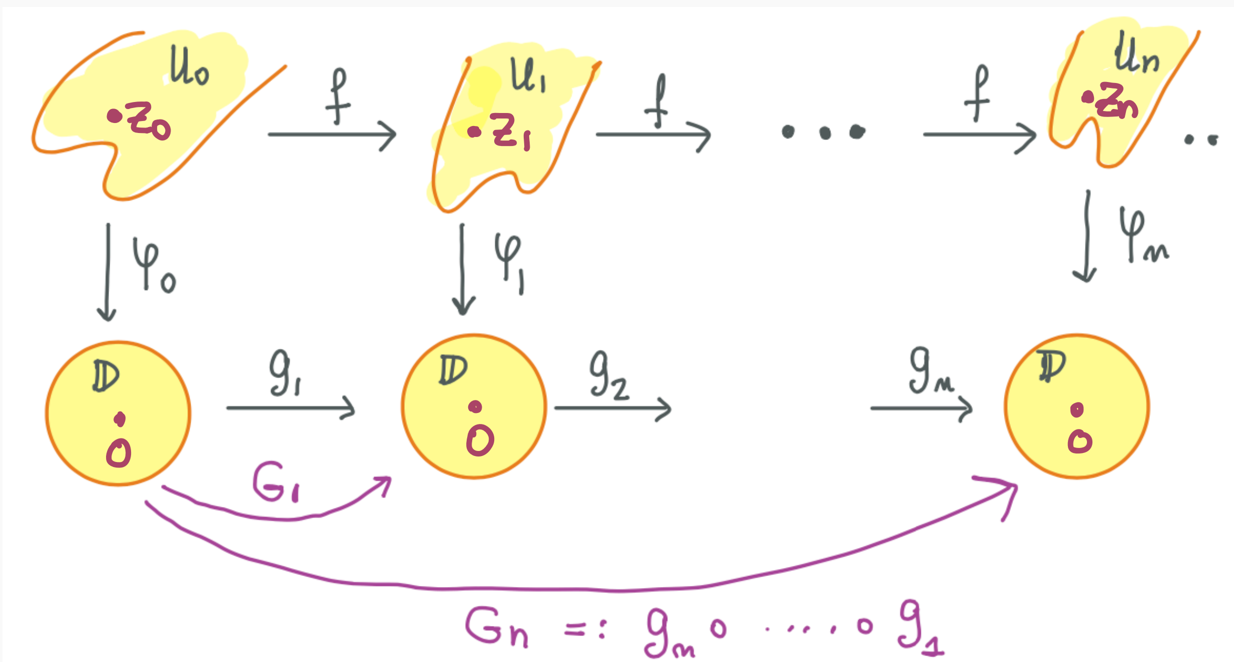

- (g_n) sequence of inner functions **associated to** to $(f|_{U_n})$.
- We say that $(f|_{U_n})$ and (g_n) are **conformally equivalent**.

ASSOCIATED SEQUENCES OF INNER FUNCTIONS

QUESTION: Can every **sequence** of inner functions be **realized** by the wandering orbit of some entire map f? If so, what can we say about f?

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ANSWER: Yes, under some conditions.

Theorem [EFGP'24]
Let $(\widetilde{b}_n)$ sequence of **uniformly hyperbolic** finite Blaschke products. Then, $\exists$ an entire transcendental function f with an orbit of scwd (simply connected wandering domains) (U_n) and a sequence of conformal maps $(\Theta_n : U_n \to \mathbb{D})$ such that

$$f|_{U_n} = \Theta_{n+1}^{-1} \circ \widetilde{b}_n \circ \Theta_n.$$

UNIFORMLY HYPERBOLIC BLASCHKE PRODUCTS

Consider Blaschke products: $b(z) := z \prod_{k=1}^{d-1} \frac{z - a_k}{1 - \overline{a_k} z}$

Definition/Theorem

Let (b_n) be a sequence of Blaschke products as above with $2 \leq \deg(b_n) = d_n \leq d < \infty$. Then TFAE:

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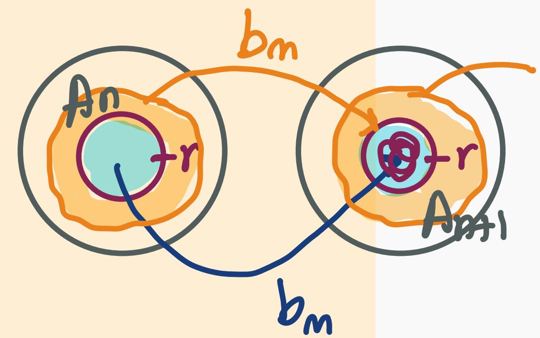

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- Surgery in a **finite number of wandering domains** is "easy" and has been done before [F-Henriksen'09] [Evdoridou-Rempe-Sixsmith'20]

QUASICONFORMAL SURGERY (crash course)

Definition

A map $\phi : U \to V$, $U, V \subset \mathbb{C}$ is $K-$**quasiconformal** if :

1. ϕ is an orientation preserving homeomorphism,
2. ϕ is absolutely continuous on lines ($\Rightarrow$ differentiable a.e.)
3. $||\mu(z)||_\infty \leq k < 1$ a.e. where $K := \frac{1+k}{1-k} \geq 1$ and

$$\mu(z) := \phi^* \mu_0(z) := \frac{\partial_{\bar{z}}\phi}{\partial_z \phi}(z) \text{ a.e. where defined.}$$

- 1-qc maps are conformal ($\mu = \mu_0 \equiv 0,\ k = 0$).
- $K \in [1, \infty)$ (or $k \in [0, 1)$) measures **angle distortion**, or how far ϕ is from being conformal.
- μ encodes the information of the **field of ellipses** $(D_z\phi)^{-1}(\mathbb{S}^1)$ (eccentricity and orientation), defined a.e.

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$g:U\to V$ is a $K-$**quasiregular** map if g is locally $K-$quasiconformal except at a discrete set of points.

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- Quasiregular maps often are used as **dynamical models** of holomorphic maps, i.e. $\phi\circ g = G\circ\phi$, where g is q.r., ϕ is q.c. and G is holomorphic. We write $g \underset{qc}{\sim} G$.

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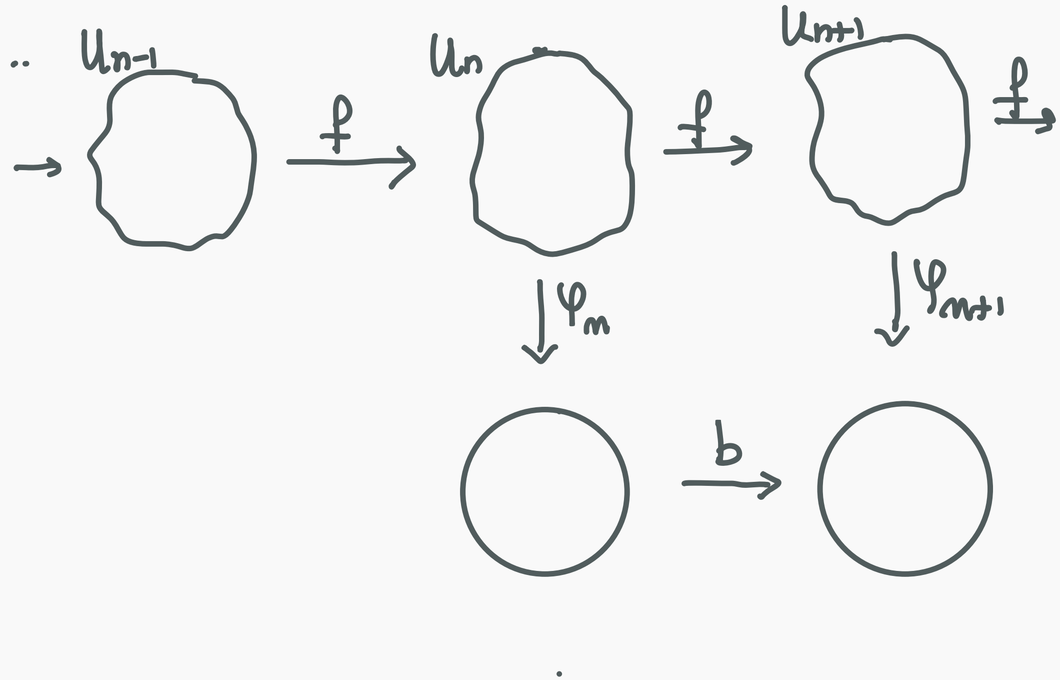
The proof is based on the celebrated theorem of Morrey, Ahlfors, Bers, and Bojarski (the **Measurable Riemann Mapping Theorem**) which proves the existence of $K-$quasiconformal solutions of the Beltrami PDE

$$\partial_{\bar{z}}\phi = \mu(z)\,\partial_z\phi$$

. for every μ measurable such that $||\mu||_\infty \leq k < 1$.

Modifying ONE inner function



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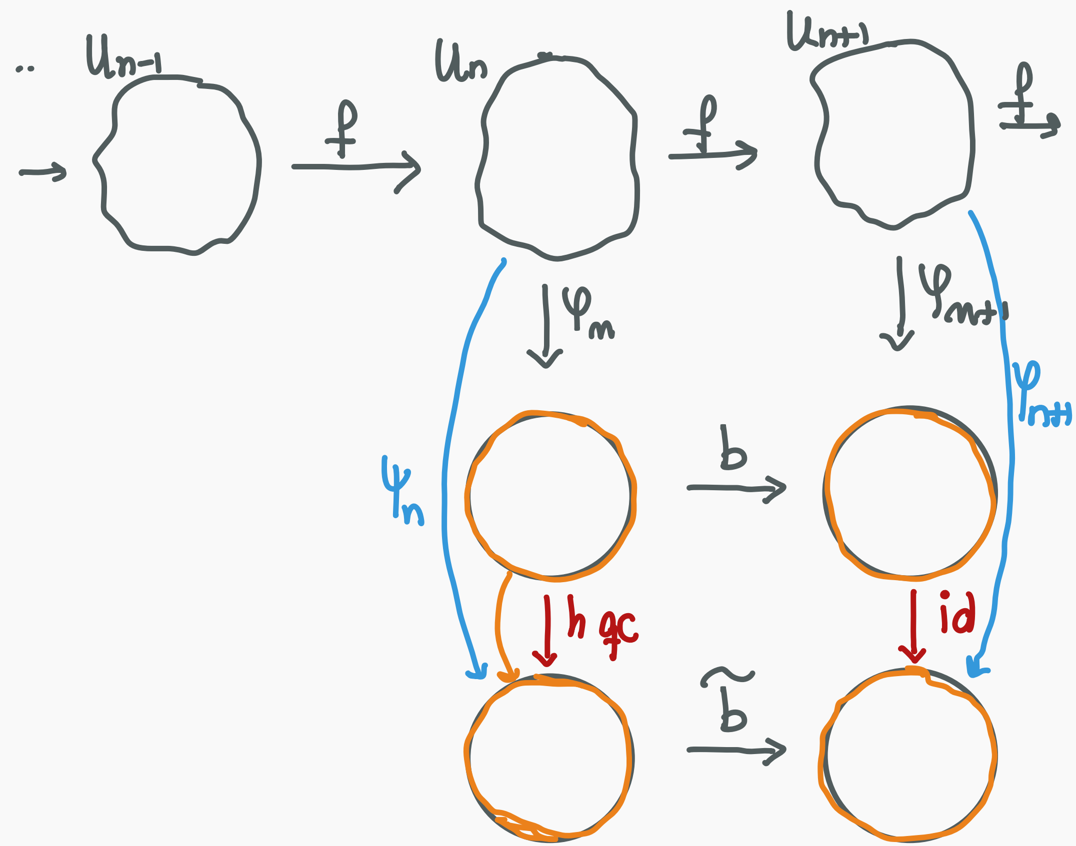

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New map (qr):

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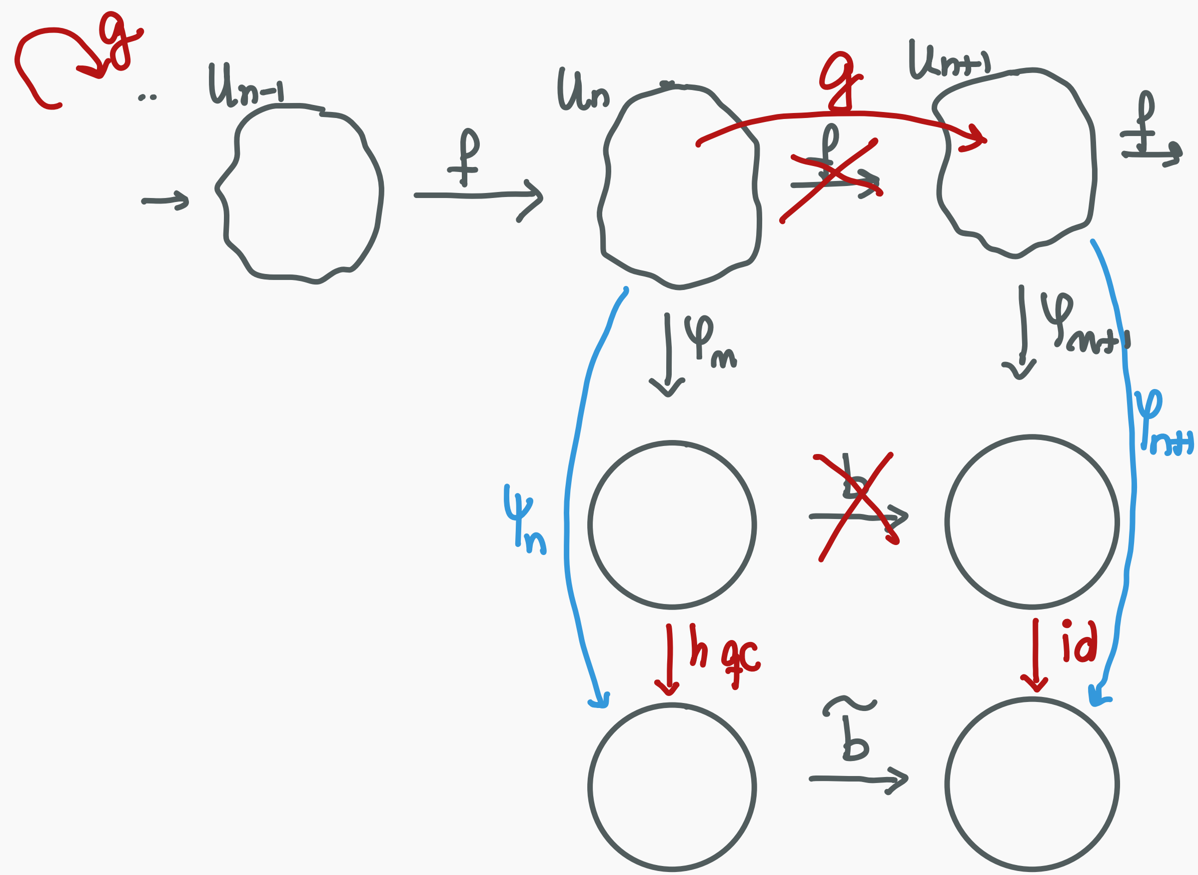

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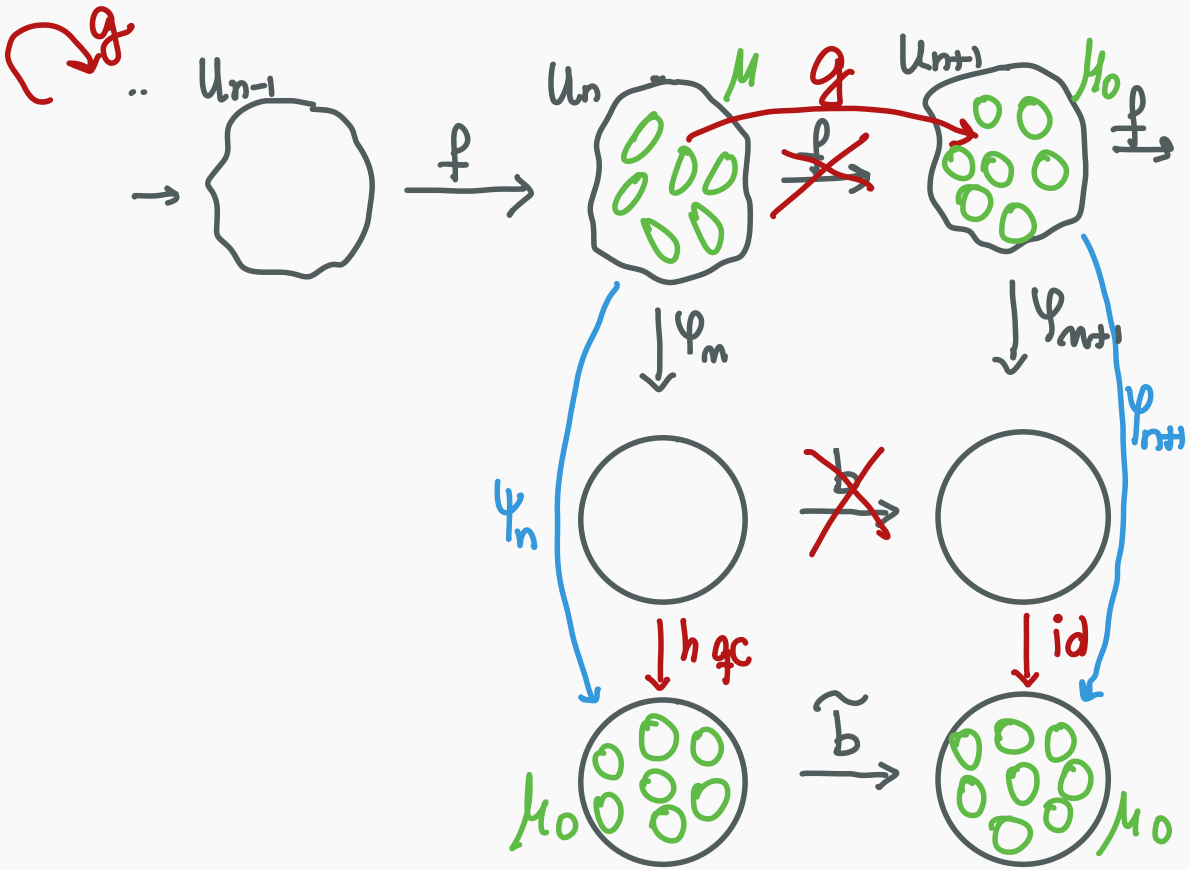

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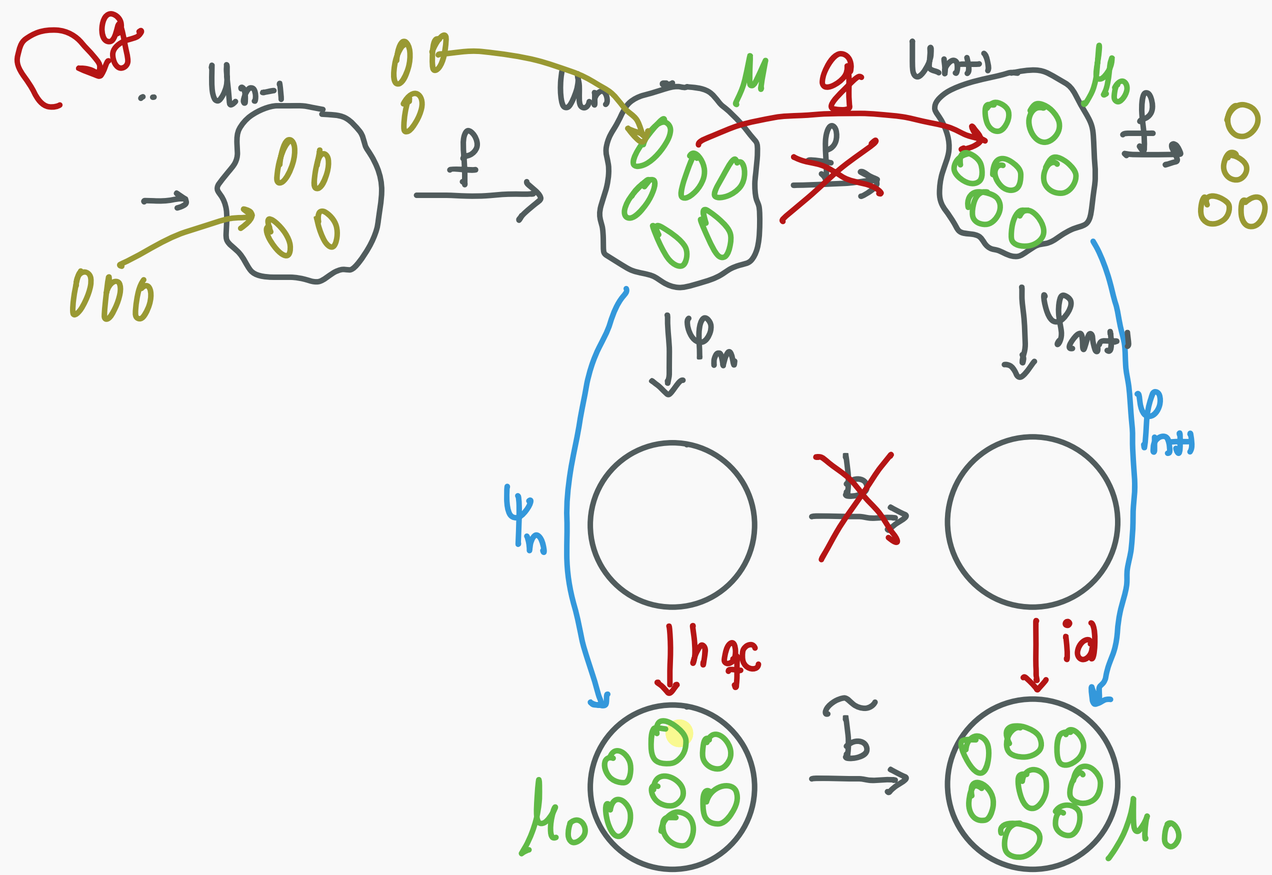

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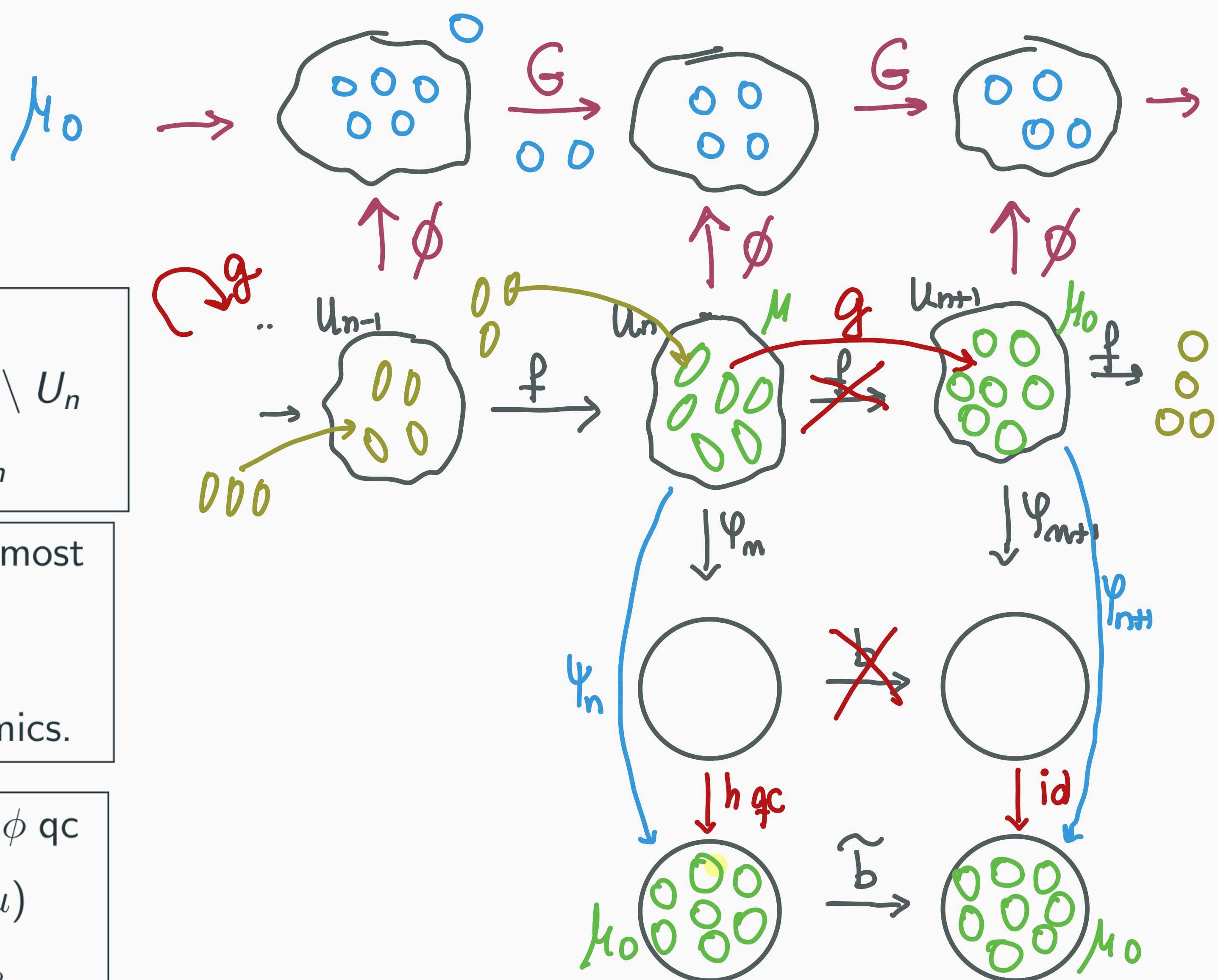
$$\begin{array}{ccc} (\mathbb{C},\mu) & \xrightarrow{g(qr)} & (\mathbb{C},\mu) \\ \downarrow \phi & & \downarrow \phi \\ (\mathbb{C},\mu_0) & \xrightarrow{G(hol)} & (\mathbb{C},\mu_0) \end{array}$$



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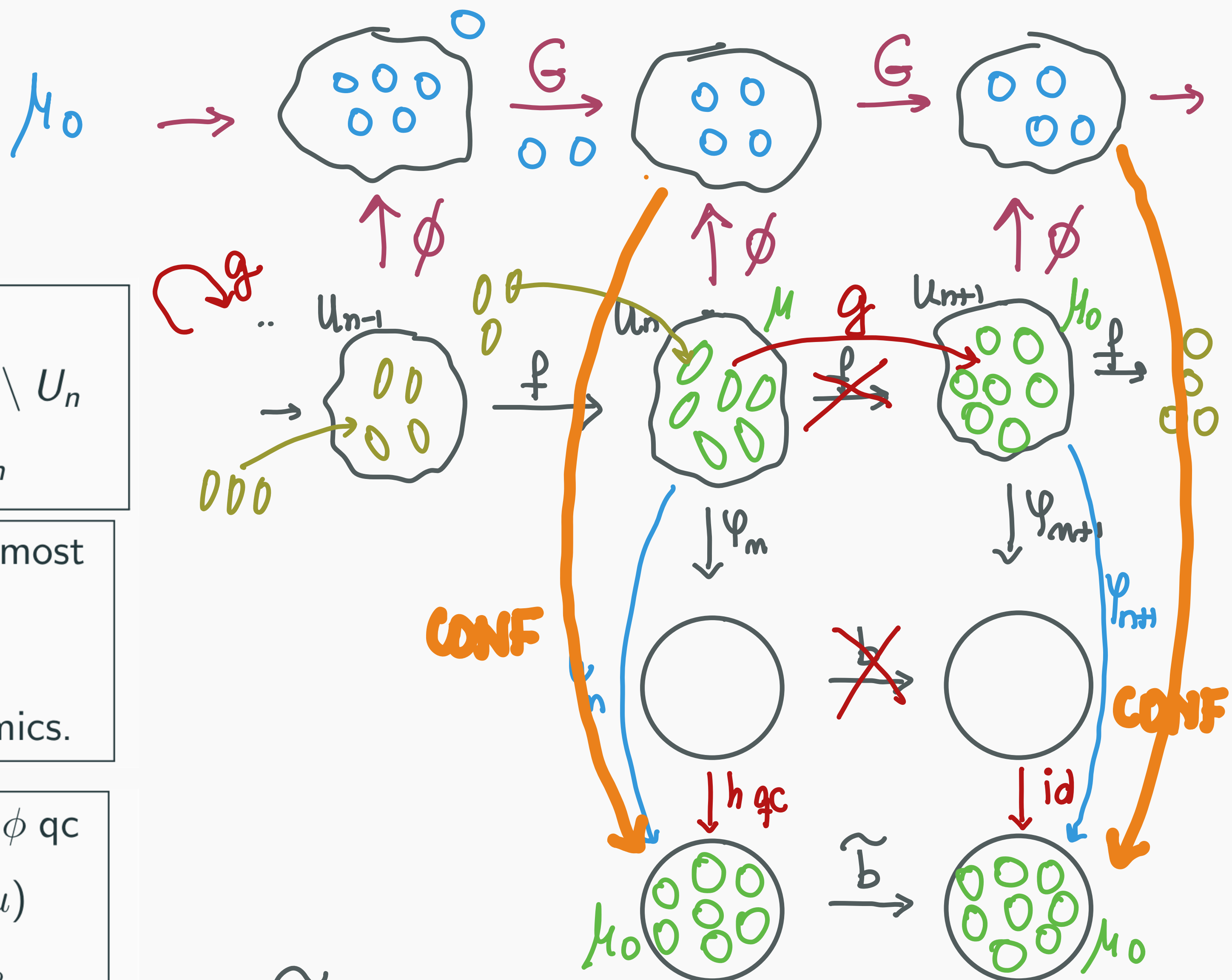

$\tilde{b}$ new inner function

RECALL:

QUESTION: Can every **sequence** of inner functions be **realized** by the wandering orbit of some entire map f? If so, what can we say about f?

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Theorem [EFGP'24]

Let $(\widetilde{b}_n)$ sequence of **uniformly hyperbolic** finite Blaschke products. Then, $\exists$ an entire transcendental function f with an orbit of scwd (simply connected wandering domains) (U_n) and a sequence of conformal maps $(\Theta_n : U_n \to \mathbb{D})$ such that

$$f|_{U_n} = \Theta_{n+1}^{-1} \circ \widetilde{b}_n \circ \Theta_n.$$

Step 1: Construction of the "gluing map"

Two **uniformly hyperbolic** sequences

$$b_n\colon \mathbb{D}^{(n)} \to \mathbb{D}^{(n+1)} \qquad \text{and} \qquad \widetilde{b}_n\colon \widetilde{\mathbb{D}}^{(n)} \to \widetilde{\mathbb{D}}^{(n+1)}$$

with **matching degrees** $d_n = \tilde{d}_n$, are $K-$**qc equivalent** near the boundary of $\mathbb{D}$.

STEP 1 of the proof: The gluing map

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More precisely:

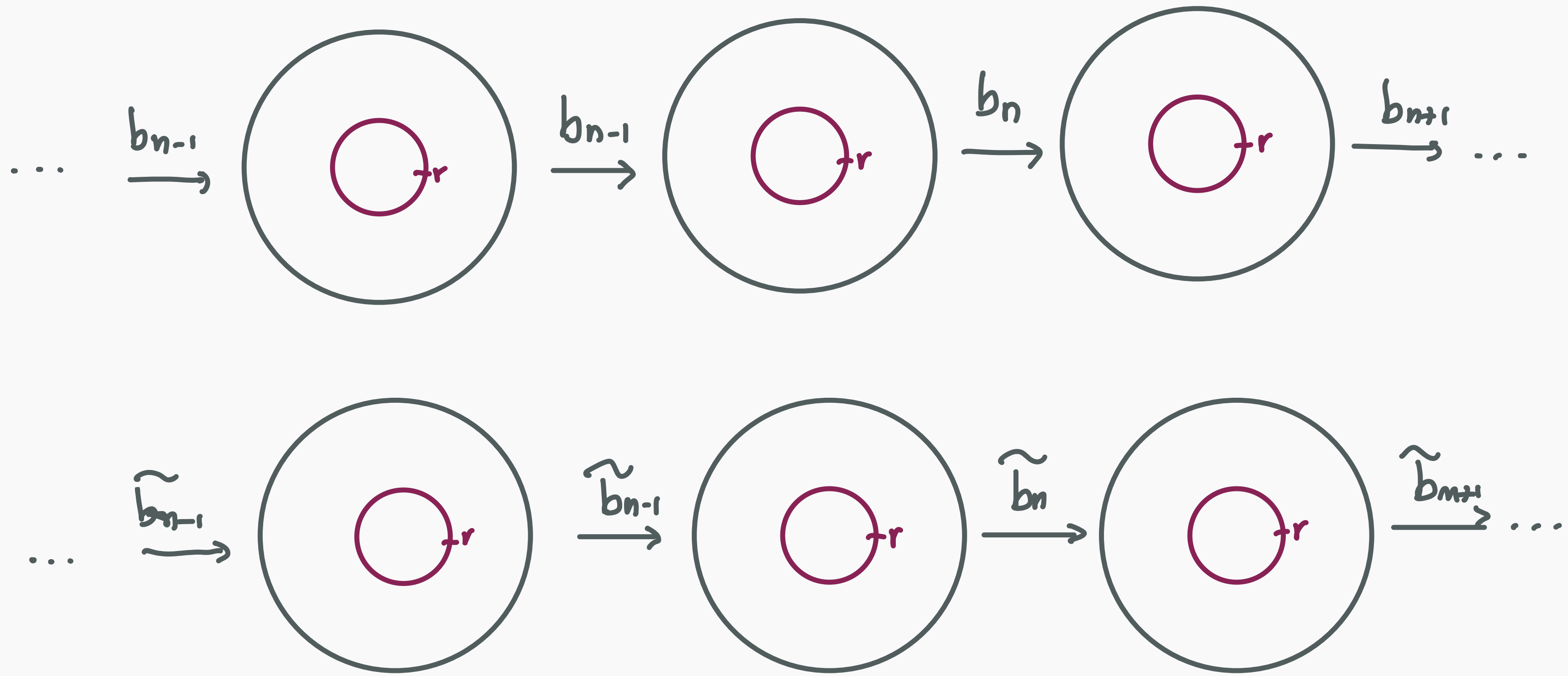
$\exists\ K > 1$ and $h_n\colon \mathbb{D}^{(n)} \to \widetilde{\mathbb{D}}^{(n)}$ $K-$qc maps, such that for all $n \in \mathbb{N}$

(a) $h_n(z) = z$ for $|z| \leq r$;

(b) $b_n = h_{n+1}^{-1} \circ \widetilde{b}_n \circ h_n$ on the annulus $C_n := \mathbb{D}^{(n)} \setminus b_n^{-1}(\mathbb{D}_r)$.

$$\begin{array}{ccccccc}
\dots & \xrightarrow{b_{n-1}} & \mathbb{D}^{(n)} & \xrightarrow{b_n} & \mathbb{D}^{(n+1)} & \xrightarrow{b_{n+1}} & \dots \\
 & & \Big\downarrow{\scriptstyle h_n} & & \Big\downarrow{\scriptstyle h_{n+1}} & & \\
\dots & \xrightarrow{\widetilde{b}_{n-1}} & \widetilde{\mathbb{D}}^{(n)} & \xrightarrow{\widetilde{b}_n} & \widetilde{\mathbb{D}}^{(n+1)} & \xrightarrow{\widetilde{b}_{n+1}} & \dots
\end{array}$$

The gluing map



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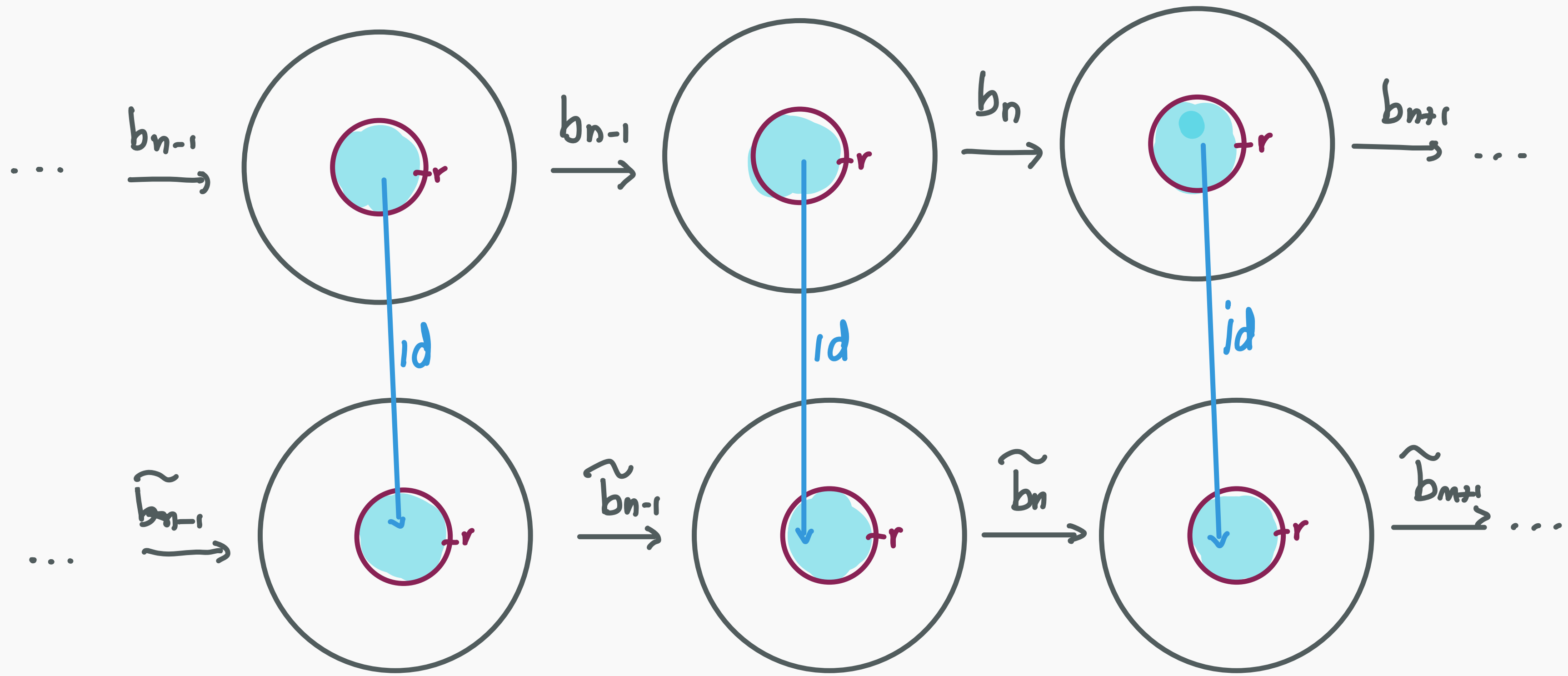

$$h_n = \begin{cases} \mathrm{Id} & \text{on } \mathbb{D}_r \\ & \\ & \end{cases}$$

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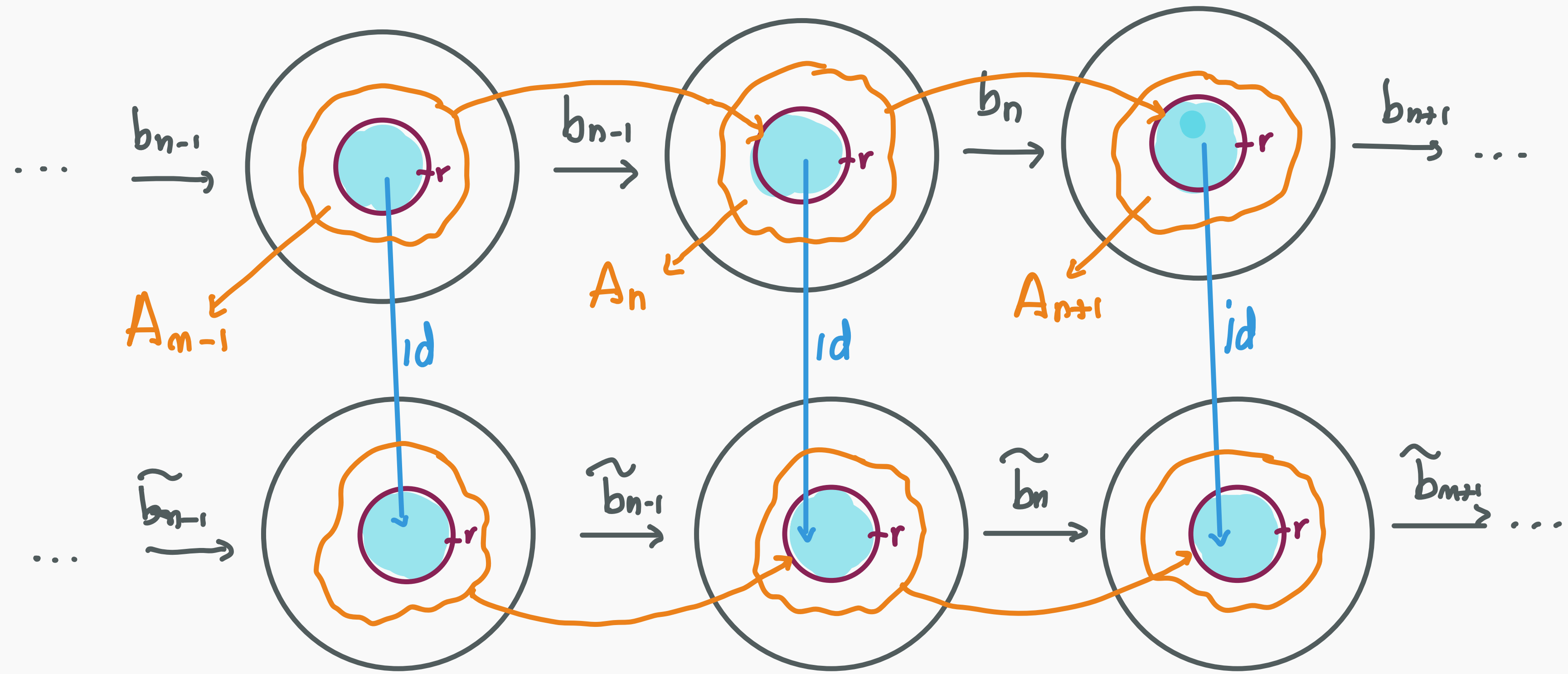

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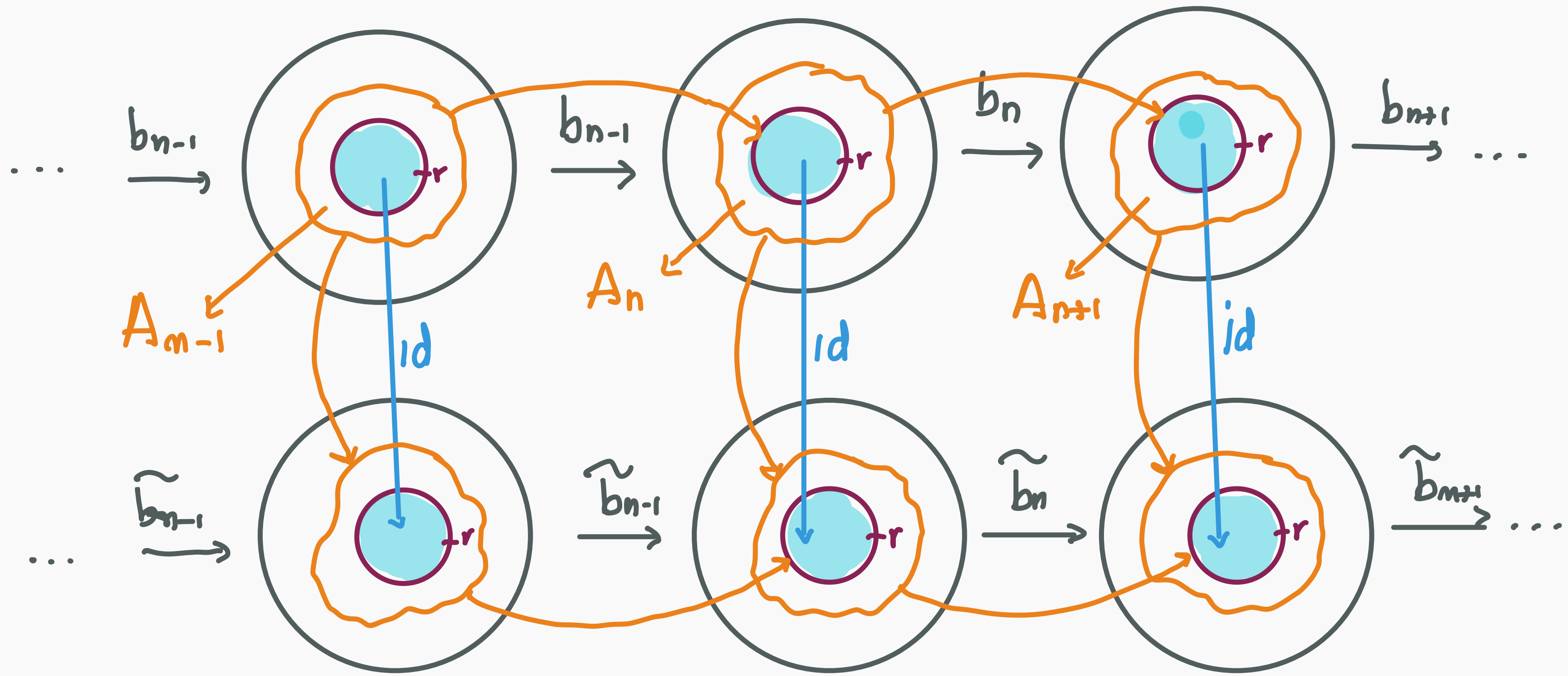

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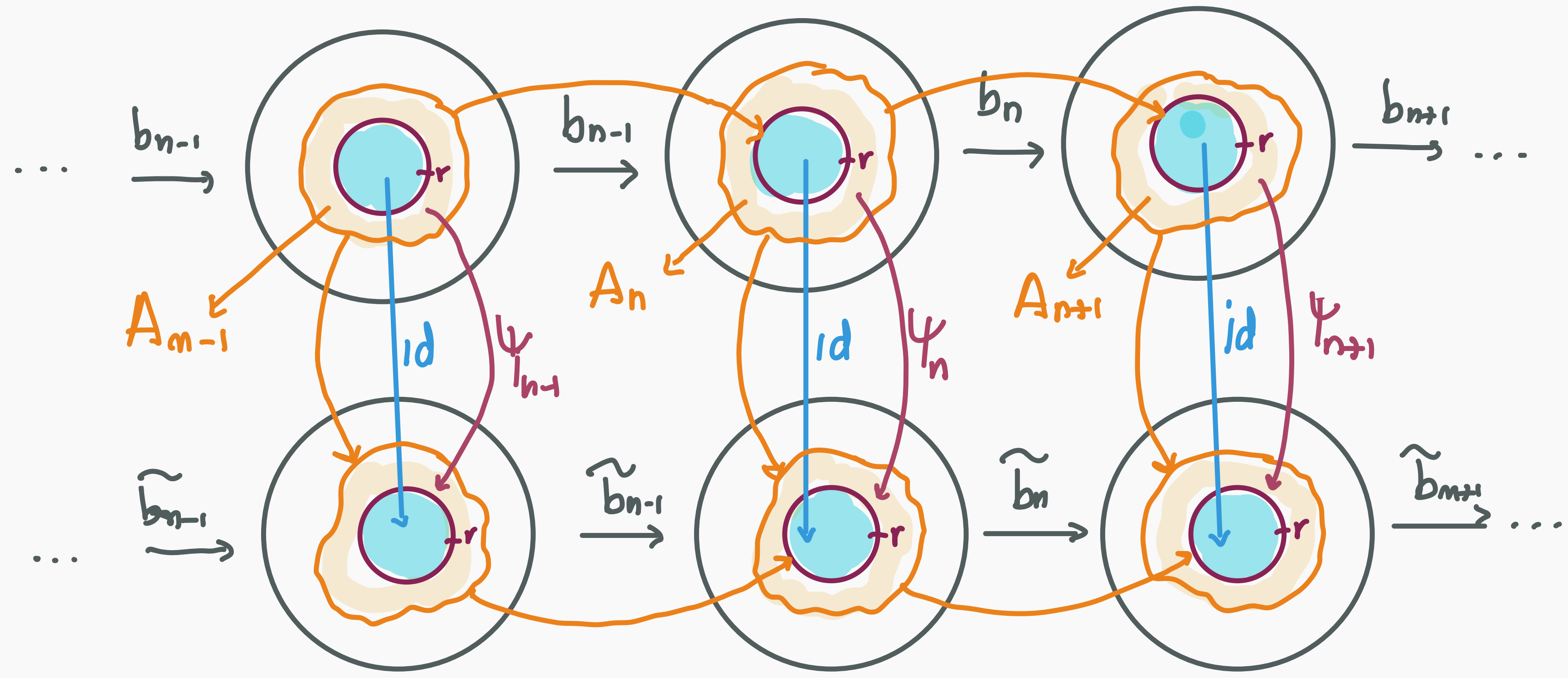

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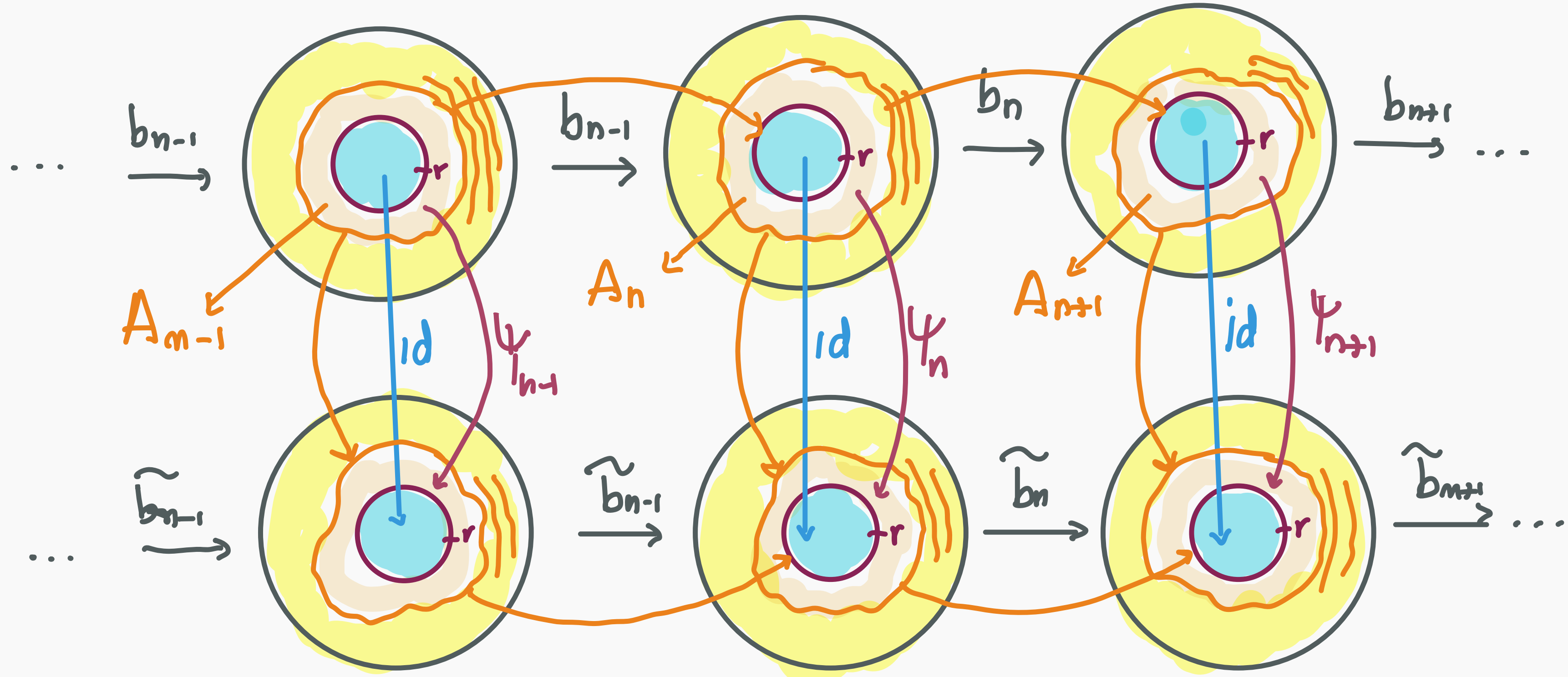

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(one annulus at a time)

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STEP 2 of the proof: A replacement surgery procedure

Step 2: Gluing the sequence inside an actual wandering domain

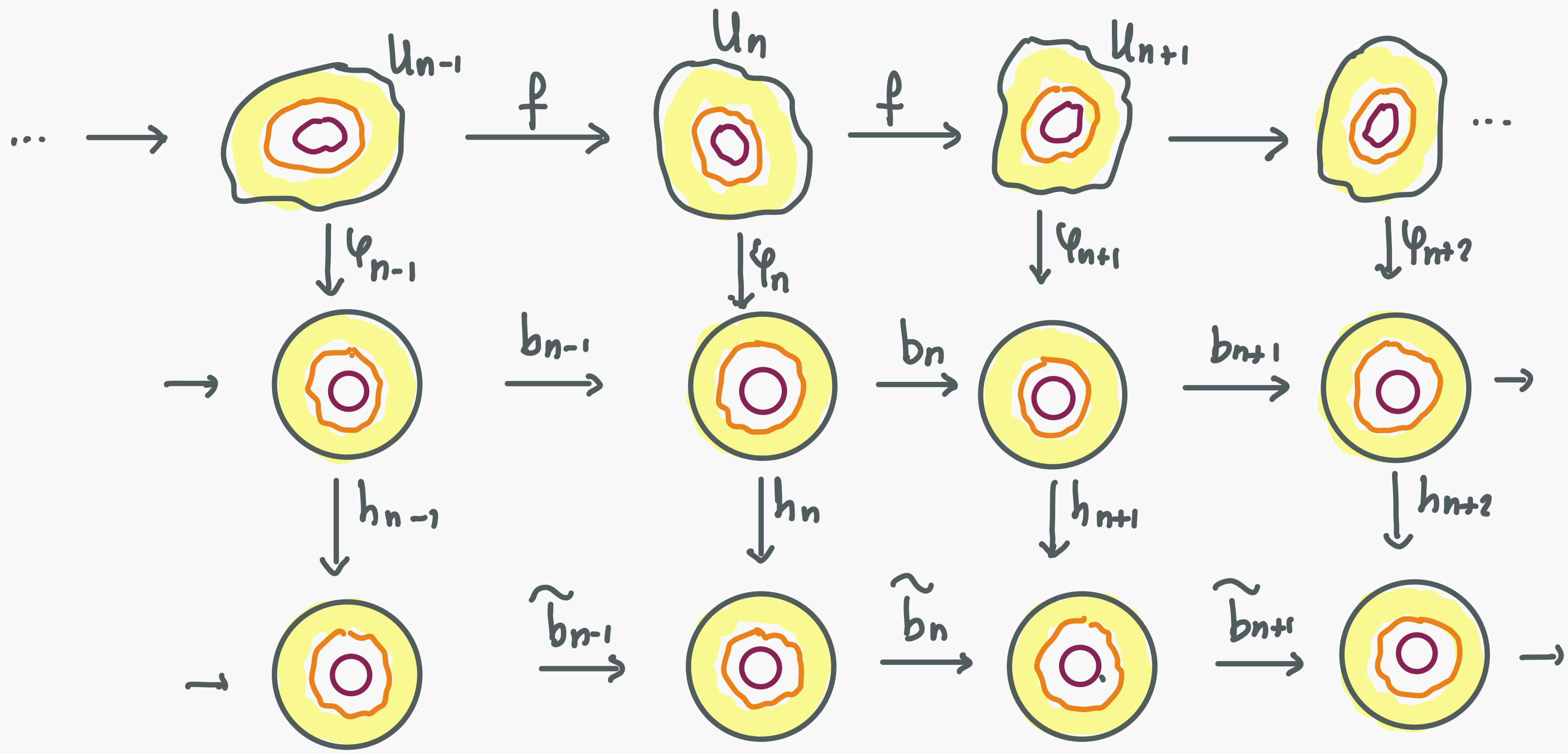
Theorem [EFGP'24]

Let f be an entire function, (U_n) an orbit of scwd, and $(\widetilde{b}_n : \mathbb{D} \to \mathbb{D})$ a sequence of Blaschke products such that $2 \leq \deg(f|_{U_n}) = \deg(\widetilde{b}_n)$ for all $n \geq 0$. Suppose both sequences $(f|_{U_n})$ and $(\widetilde{b}_n)$ are uniformly hyperbolic.
Then $\exists$ F entire with $(\widetilde{U}_n)$ scwd, such that:

(a) There is a sequence of conformal maps $\theta_n : \widetilde{U}_n \to \mathbb{D}$ such that
$$F|_{\widetilde{U}_n} = \theta_{n+1}^{-1} \circ \widetilde{b}_n \circ \theta_n.$$

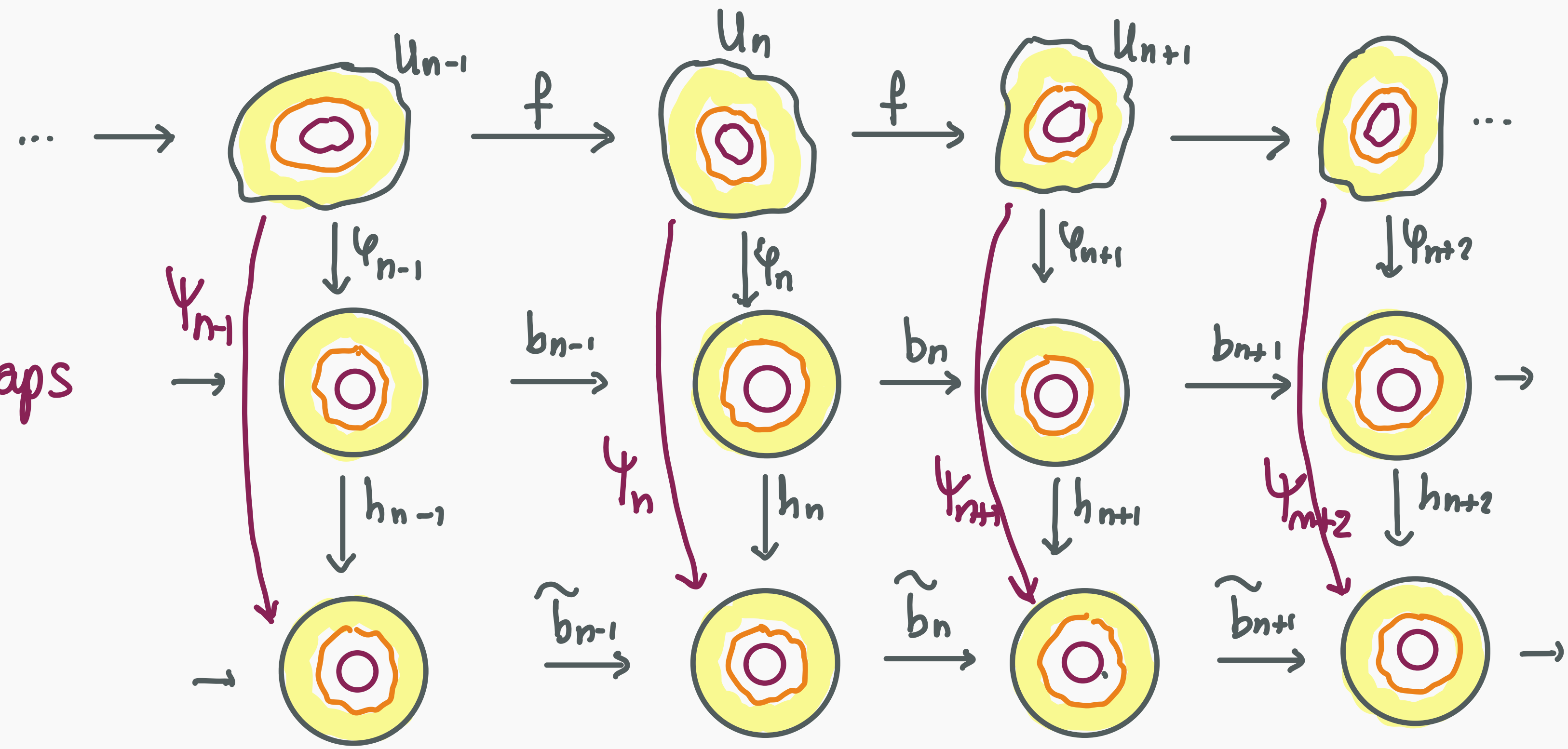
(b) f and F are **qc conjugate** outside the wandering orbits (and beyond).

STEP 2: A replacement surgery procedure



STEP 2: A replacement surgery procedure

Ψ_n gluing maps



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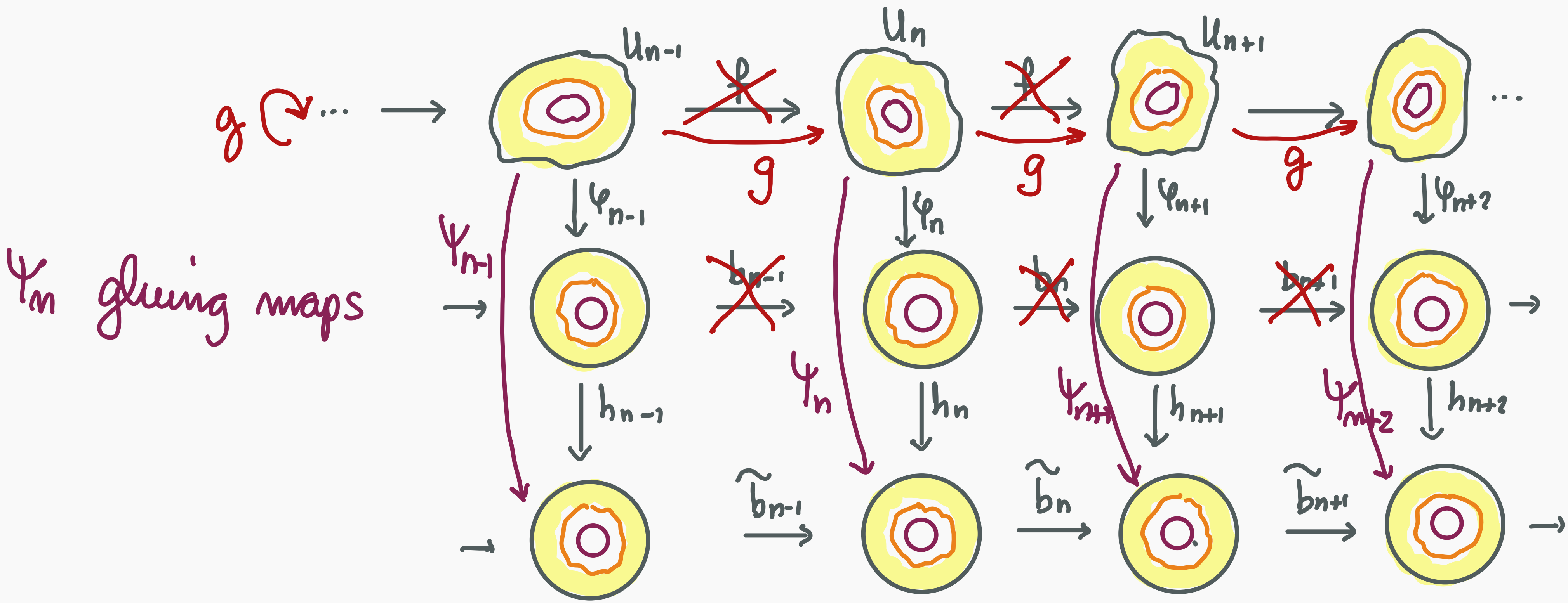

New map (unif. $K-$ qr):

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OBS: $g = f$ near ∂U_n (yellow)

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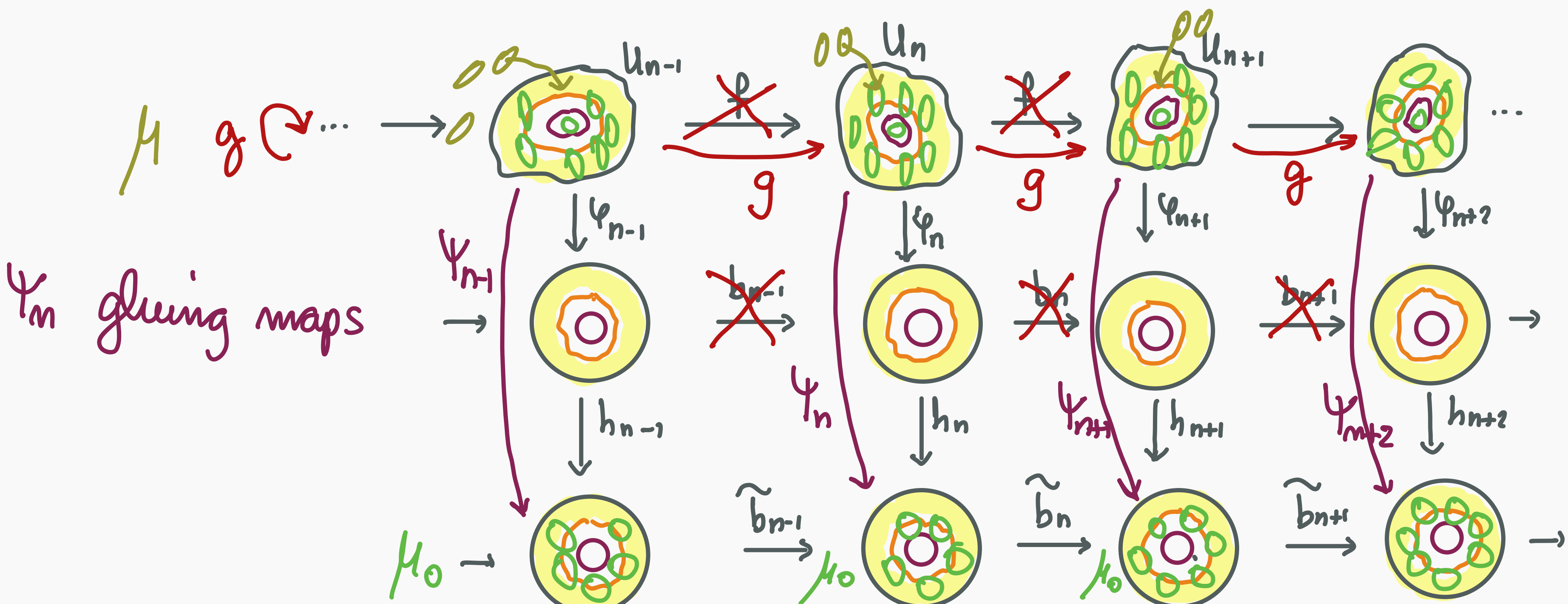

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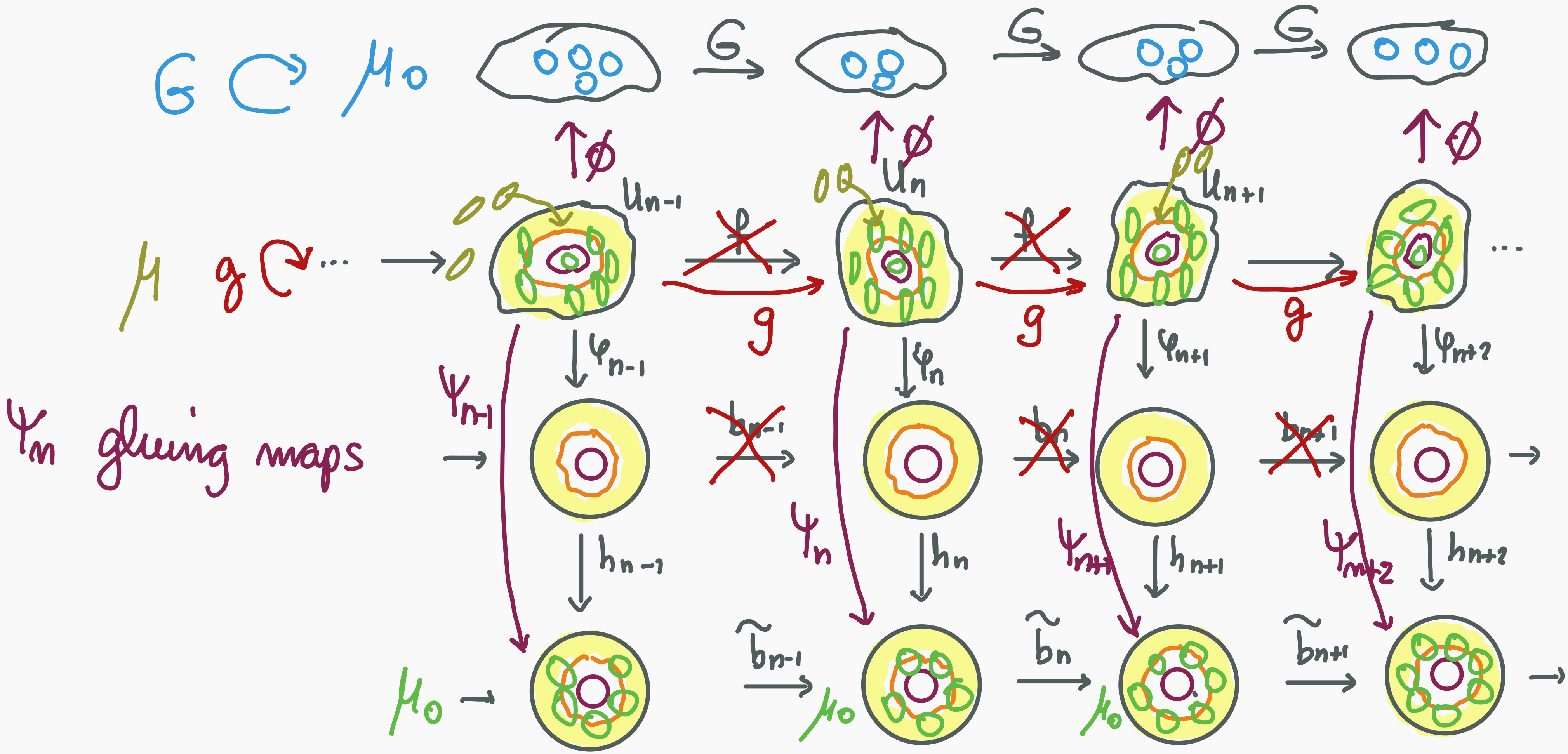

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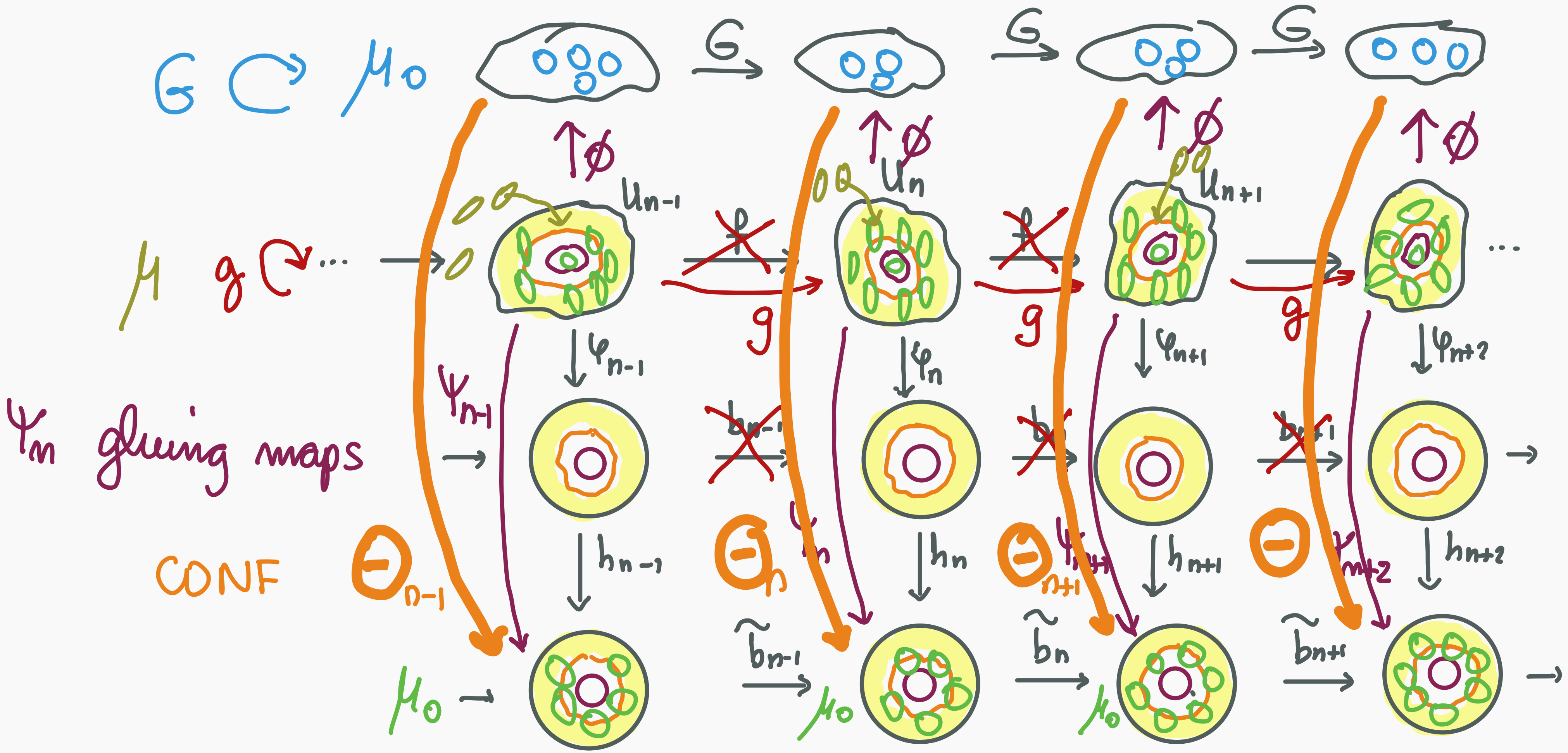

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$\{\widetilde{b}_n\}$ NEW ASSOC. SEQUENCE

Final step: Finding the template

Step 3: Finding the template f where to glue $(\widetilde{b_n})$

Lemma [BERFS'22]

Let (d_n) be a sequence in $\mathbb{N}$, with $2 \leq d_n \leq d < \infty$. Then $\exists$ an entire function f with an orbit of scwd (U_n) such that $(f|_{U_n})$ is uniformly hyperbolic and $\deg(f|_{U_n}) = d_n$ for all n.

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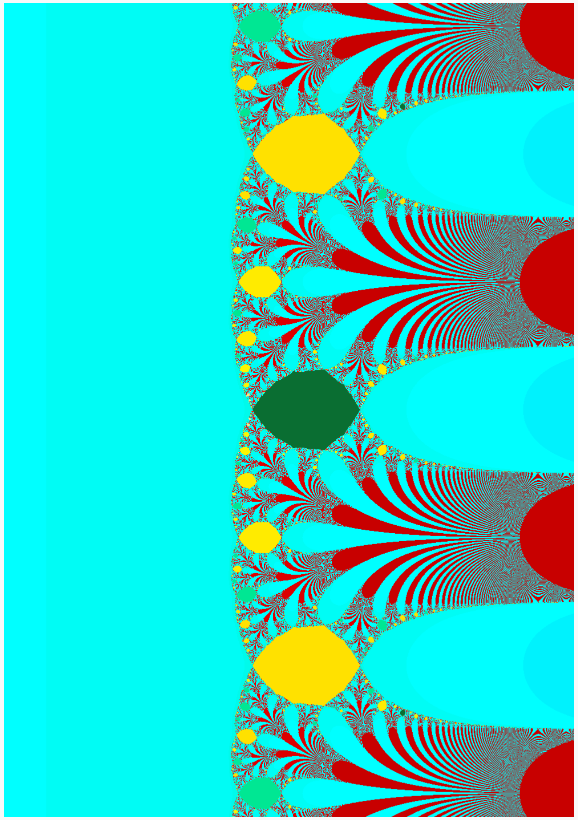
But if $d_n = d$ for all $n \in \mathbb{N}$, then f can be written explicitely [ERS'20] (important for our applications).



FURTHER MOTIVATION / APPLICATIONS

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- We first construct a sequence of (unif. hyp) **Blaschke products** of constant degree $d \geq 2$, showing the desired property. Then we use the **realizability theorem**. The **control** on the final map f allows us to show the coexistence property.

THANK YOU FOR YOUR ATTENTION!