L^2-estimates for $\bar{\partial}$ and ODEs

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- $|\alpha|^2_{i\partial\bar\partial\varphi} \leq H \iff i\bar\alpha \wedge \alpha \leq H\, i\partial\bar\partial\varphi$
- The original Hörmander's formulation with $2|\alpha|^2/c$, where $(\varphi_{j\bar k}) \geq c(\delta_{jk})$, instead of $|\alpha|^2_{i\partial\bar\partial\varphi}$. The above formulation for (0,1)-forms is due to Demailly.

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- If $\partial\Omega \in C^\infty$, strongly convex, then $G \in C^\infty(\overline{\Omega} \setminus \{0\})$ (Lempert)

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$$i\bar\alpha \wedge \alpha = (\chi' \circ G)^2\, i\partial G \wedge \bar\partial G,$$

$$i\partial\bar\partial\varphi \ge \frac{1}{(1-G)^2} i\partial G \wedge \bar\partial G,$$

hence

$$i\bar\alpha \wedge \alpha \le (1-G)^2 (\chi' \circ G)^2\, i\partial\bar\partial\varphi.$$

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- If $n = 1$ then $e^{2t}/\lambda(\{G < t\}) \to (c_\Omega(0))^2/\pi$ as $t \to -\infty$, where

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- $c(n,t)/\lambda(\{G < t\}) \to 0$ as $t \to -\infty$

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- Using Lempert's theory one can show that if Ω is convex then $I_\Omega^A = I_\Omega^K$, where

$$I_\Omega^K := \{\varphi'(0) \colon \varphi \in \mathcal{O}(\mathbb{D}, \Omega),\ \varphi(0) = 0\}$$

is the Kobayashi indicatrix.

ODE Question $\lim_{t\to-\infty} e^{-t}c(t) > 0$, where

$$c(t) := \sup\{\left(\int_{-\infty}^{t} \sqrt{\gamma''(s)e^{\gamma(s)}}e^{s}ds\right)^2 : \gamma \in CVX \cap C^2(\mathbb{R}_-, \mathbb{R}_-)\}.$$

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- (Thomas) If Ω is a disc not centered at 0 then $\log \lambda(\{G < t\})$ is not convex.

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Monotonicity

Question: Is $e^{2nt}/\lambda(\{G<t\})$ decreasing to $1/\lambda(I_\Omega^A)$ as $t\to-\infty$?

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By the co-area formula

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Proposition If Ω is convex then

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Lemma If Ω is convex, $\partial\Omega \in C^1$ then for $w \in \partial\Omega$

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Remark The constant 2 cannot be improved (disc).

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If u is a solution to $\bar\partial u = \alpha$ minimal in $L^2(\Omega, e^{-\psi/2-\varphi})$ then $u \perp \ker \bar\partial$ and

$$\int_\Omega u\bar f \, e^{-\psi/2-\varphi} d\lambda = 0, \quad f \in \ker \bar\partial.$$

This means that $v := e^{\psi/2}u$ is a solution to

$$\bar{\partial}v = \beta := \bar{\partial}(e^{\psi/2}u) = \left(\alpha + \frac{1}{2}u\bar{\partial}\psi\right)e^{\psi/2}$$

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- This estimate, together with an ODE, can give Suita and Ohsawa-Takegoshi with the constant $1.95388\ldots$ (earlier obtained by Guan-Zhou).

B.: If $H := |\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \leq 1$ in Ω and $H \leq a < 1$ on $\operatorname{supp}\alpha$ then

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where $t > 0$. May take $0 \leq \mu < 1$ s.th. $1 - (1+t)H = \mu(1-H)$.

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- If $H \leq a < 1$ on $\operatorname{supp}\alpha$ then for $\mu = 1/(1+\sqrt{a})$ we recover the previous result.

Thank you!