Superforms, convex geometry and valuations.

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Background.

We use coordinates $x = x_1, ...x_n$ on $\mathbb{R}^n$ and $x + i\xi = x_1 + i\xi_1, ...x_n + i\xi_n$ on $\mathbb{C}^n$. A superform on $\mathbb{R}^n$ is a form

$$\alpha = \sum \alpha_{IJ}(x) dx_I \wedge d\xi_J$$

on $\mathbb{C}^n$ whose coefficients depend only on x. If all $|I| = p$, $|J| = q$, α has bidegree (p, q). If $p = q = n$, its superintegral is

$$\int \alpha = \int a(x) dx \wedge d\xi := \int a(x) dx.$$

We can also allow the α_{IJ} to be measures. Then we get supercurrents, dual to the space of continuous superforms.

The ordinary exterior derivative, d, acts on superforms and -currents. We let

$$d^{\#} = \sum \frac{\partial \alpha_{IJ}}{\partial x_i} d\xi_i \wedge dx_I \wedge d\xi_J (= d^c \alpha).$$

If $\psi(x)$ is a function

$$dd^{\#}\psi = \sum \psi_{ij} dx_i \wedge \xi_j.$$

ψ is convex if and only if $dd^{\#}\psi \geq 0$. A (p, p)-current

$$\sum \Omega_{IJ} dx_I \wedge d\xi_J$$

is symmetric if $\Omega_{IJ} = \Omega_{JI}$. There is a notion of positivity for symmetric currents too.

Ω is symmetric if and only if $J(\Omega) = \Omega$, where J is the complex structure on $\mathbb{C}^n$.

Bedford-Taylor

Let Ω be closed, symmetric and positive. The following consequence of the theory of Bedford and Taylor is very useful:

Theorem

If ψ is convex, then

$$dd^{\#}\psi \wedge \Omega$$

is again well defined closed symmetric and positive. It depends on ψ continuously.

Corollary

$$(dd^{\#}\psi)^n/n! =: MA(\psi)$$

is a well defined positive measure. It depends on ψ continuously.

It coincides with Alexandrov's definition of the Monge-Ampere measure.

Volumes.

Let K be a convex body and

$$h_K(x) = \sup_{y \in K} x \cdot y$$

its support function. It is 1-homogeneous and convex.

Theorem

$$MA(h_K) = c_n |K| \delta_0,$$

where δ_0 is a unit Dirac measure at the origin and $|K|$ is the volume of K.

Mixed volumes

If $K_1, ...K_n$ are convex bodies and $t = (t_1, ...t_n), t_i \geq 0$, let

$$K_t = t_1 K_1 + ...t_n K_n = \{\sum t_i x_i; x_i \in K_i\}$$

(Minkowski sum of K_i). Notice that

$$h_{K_t} = t_1 h_{K_1} + ...t_n h_{K_n}.$$

It follows that the volume of K_t is a polynomial in t. The coefficient of $t_1...t_n$ divided by $n!$ is the mixed volume

$$V(K_1, ...K_n).$$

It is a polarization of the volume in the sense that it is Minkowski linear in each slot and $V(K, ...K) = |K|$.

Thus the mixed volumes are

$$V(K_1, ...K_n) = \int dd^{\#} h_{K_1} \wedge ...dd^{\#} h_{K_n} / n!,$$

where h_{K_i} are the support functions of K_i.

An inequality

We will deal mostly with convex functions of linear growth, $\phi \le A|x| + B$. The indicator of such a function is

$$\phi^\circ(x) = \lim_{t \to \infty} \phi(tx)/t.$$

The indicator is 1-homogeneous; hence the support function of some convex body.

Proposition

If $\phi_i^\circ \le \psi_i^\circ$, then

$$\int dd^\# \phi_1 \wedge ... dd^\# \phi_n \le \int dd^\# \psi_1 \wedge ... dd^\# \psi_n.$$

In particular, such integrals depend only on the indicators of the functions.

Alexandrov-Fenchel

It follows that the mixed volumes are

$$V(K_1, ...K_n) = \int dd^{\#}\psi_1 \wedge ...dd^{\#}\psi_n/n!,$$

where ψ_i are any convex functions with indicators h_{K_i}.

Theorem

Fix $K_3, ...K_n$. Then

$$V(K, L, K_3, ...K_n)^2 \geq V(K, K, K_3, ...K_n)V(L, L, K_3, ...K_n).$$

This is the Alexandrov-Fenchel inequality, a wrong-way Cauchy inequality.

Valuations

A real valued valuation on convex bodies is a map

$$K \to \Theta(K) \in \mathbb{R},$$

satisfying $\Theta(\emptyset) = 0$ and

$$\Theta(K \cup L) + \Theta(K \cap L) = \Theta(K) + \Theta(L),$$

if $K \cup L$ is convex. Here are two examples:

$$\Theta_n(K) = |K| = Vol(K),$$

the ordinary Lebesgue measure, and

$$\Theta_0(K) = 1,$$

if K is nonempty.

Important fact

A valuation extends to a finitely additive measure on the Boolean algebra generated by all convex bodies.

The theory of valuations turns out to have surprising connections to otrher areas of mathematics, like topology, representation theory, Kahler geometry...

Example: Take the trivial valuation; $\Theta(K) = 1$ if $K$is nonempty and extend it to a finitely additive measure. Then

$$\Theta(A) = \chi(A),$$

the Euler characteristic of A.

This is because

$$\chi(A \cup B) + \chi(A \cap B) = \chi(A) + \chi(B),$$

as follows from the Mayer-Vietoris sequence

$$0 \to \mathcal{E}(A \cup B) \to \mathcal{E}(A) \oplus \mathcal{E}(B) \to \mathcal{E}(A \cap B) \to 0.$$

To see more examples we need to find more valuations. Here is one important way to construct new valuations from old ones.

New valuations from old ones

If A is a convex body

$$\tau_A(\Theta)(K) := \Theta(K + A),$$

the translate of Θ by A, is again a valuation. Apply this to

$$\Theta_n(K) = |K| = \int (dd^{\#} h_K)^n / n! = \int \omega_K^n / n!.$$

Then

$$\Theta_n(K + A) = \int (\omega_K + \omega_A)^n / n! = \sum \int \omega_K^k \wedge \omega_A^{n-k} / k!(n-k)!.$$

It follows that

$$K \to \int \omega_K^k \wedge \omega_A^{n-k}$$

is a valuation for any convex body A.

Hence...

all mixed volumes

$$K \to \int \omega_K^k \wedge \omega_{A_1} \wedge ...\omega_{A_n} = V(K[k], A_1, ...A_{n-k})$$

are valuations.
These valuations are translation invariant

$$\Theta(K + a) = \Theta(K).$$

We assume from now that all valuations we look at are translation invariant and *continuous* – in some sense.
Alesker introduced the stronger notion of

smooth valuations

Let $g \in GL(\mathbb{R}^n)$, and put

$$g^*(\Theta)(K) = \Theta(gK).$$

A valuation Θ) is called smooth if $g \to g^*(\Theta)$ is smooth. The next theorem is due to Alesker, Knoerr, van Handel ...

Theorem

Any smooth and translation invariant valuation is a finite linear combination of mixed volumes

$$\int \omega_K^k \wedge \omega_{A_1} \wedge ...\omega_{A_{n-k}} = V(K[k], A_1, ...A_{n-k}).$$

The forms $\omega_i = \omega_{A_i}$ that appear here have three properties:

a) $d\omega_i = 0$.

b) $\delta_E \omega_i = 0$, where δ_E is contraction with the Euler vector field $E = \sum x_j \partial_j$.

c) ω_i is symmetric; i. e. $J(\omega_i) = \omega_i$.

Hence

$$\Omega_I = \omega_{A_1} \wedge ... \omega_{A_{n-k}}$$

has the same properties.

Remark

The Euler vector field $E = \sum x_j \partial_j$ generates the flow on $\mathbb{R}^n$

$$x \to F_t(x) = tx.$$

By Cartan's 'magic formula' the Lie derivative of a form along this vector field is

$$L_E = d\delta_E + \delta_E d.$$

Hence, if Ω satisfies $d\Omega = 0$ and $\delta_E \Omega = 0$, Ω is invariant under the flow:

$$F_t^*(\Omega) = \Omega.$$

Theorem/Conjecture

Theorem

Let Θ be a smooth translation invariant valuation. Then there is a unique superform

$$\Omega = \Omega_0 + ...\Omega_n,$$

where Ω_p is of bidegree (p, p), satisfying a,b,c, such that

$$\Theta(K) = \sum_{k=0}^{k=n} \int \omega_K^k \wedge \Omega_{n-k}/k!.$$

Conversely, any such Ω defines a valuation (?)

A rewrite

A nicer way to write the formula is

$$\Theta(K) = \int e^{\omega_K} \wedge \Omega =: \mathcal{F}^{-1}(\Omega)(K).$$

We think of this as an inverse Fourier transform, mapping superforms to valuations, via the 'character'

$$K \to e^{\omega_K}.$$

$\Omega = \mathcal{F}(\Theta)$. We have

$$\tau_A(\Theta)(K) = \int e^{\omega_K + \omega_A} \wedge \Omega = \int e^{\omega_K} \wedge e^{\omega_A} \wedge \Omega,$$

so

$$\mathcal{F}(\tau_A \Theta) = e^{\omega_A} \wedge \mathcal{F}(\Theta).$$

Lefschetz theorem for valuations

Let $\omega_A = dd^\# h_A$, where A is strictly convex and smoothly bounded. Define a Lefschetz operator on valuations by

$$L(\Theta) = \mathcal{F}^{-1}(\omega_A \wedge \Omega),$$

if $\Theta = \mathcal{F}^{-1}(\Omega)$.

Bernig, Kotrbatý and Wannerer proved a 'Hard Lefschetz theorem':

Theorem

The map

$$\Theta \to L^{n-2p}(\Theta)$$

is a bijection on the space of valuations of order p.

Approach in terms of superforms

Let $F_s(x) = sx$ for $s > 0$. We look at (p, p)-forms Ω such that

$$\Omega^\infty := \lim_{t \to \infty} F_t^*(\Omega)$$

exists. This is an analog of functions of linear growth. If Ω is such a form we define its (cohomology) class by

$$[\Omega] = \{\Omega'; \lim_{t \to \infty} F_t^*(\Omega') = \Omega^\infty\}.$$

Hodge theorem/conjecture

The Lefschetz map is defined in terms of $dd^{\#}h_A$, where A is strictly convex and smoothly bounded. Let ψ be a smooth strictly convex function with indicator h_A, and put

$$\omega = dd^{\#}\psi.$$

The form ω defines a Kahler structure on $\mathbb{C}^n$.

Theorem

Every class $[\Omega]$ contains exactly one harmonic representative.

This would imply the Lefschetz theorem, much like it's proved in Algebraic geometry. I can prove it for $p = 1$.

Thanks!