Horn maps of semi-parabolic Hénon maps

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Joint work with F. Bianchi

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Introduction

1. Polynomial: $f(z) = z + z^2 + O(z^3) \rightsquigarrow$ horn map $h : W \to \mathbb{P}^1$
   Ecalle-Voronin: h complete invariant for loc. analytic classification
   Lavaurs: $f_{\epsilon_n}^n \to \mathcal{L} \leftrightsquigarrow h$

2. Hénon map $F : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial automorphism
   $F(x, y) = (x + x^2 + \ldots, ay + \ldots)$, $a \in \mathbb{D}^* \rightsquigarrow h : W \to \mathbb{P}^1$
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Goal

Describe dynamical properties of h

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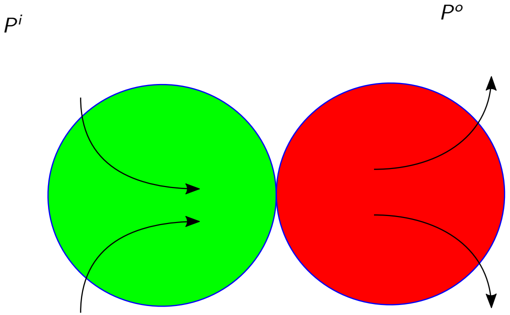
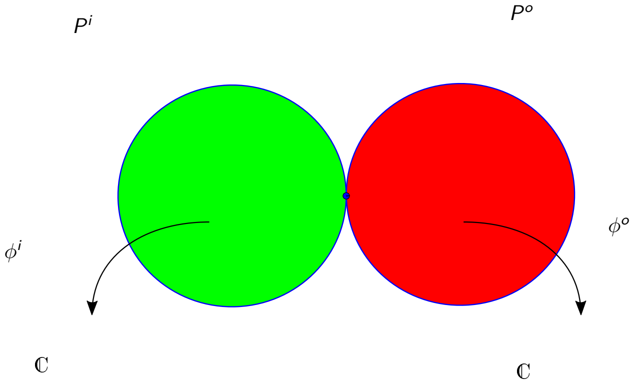
Local parabolic dynamics in dimension 1

$$f(z) = z + z^2 + O(z^3)$$



Local parabolic dynamics in dimension 1

Fatou coordinates: $\phi^{i/o} \circ f = \phi^{i/o} + 1$



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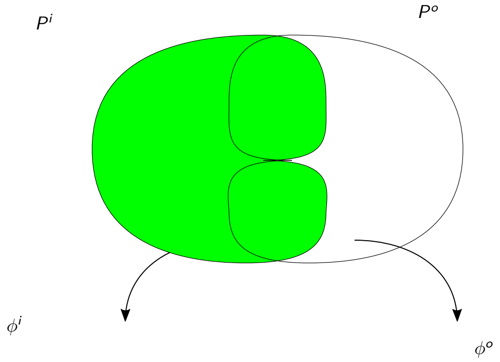

Change of Fatou coordinates: $\mathcal{L} := (\phi^o)^{-1} \circ \phi^\iota$

Semi-local theory in dimension 1

Let $f(z) = z + z^2 + O(z^3)$ be polynomial, $\mathcal{B}$ = parabolic basin

Fact

1. ϕ^ι extends to $\mathcal{B}$
2. $\psi^o := (\phi^o)^{-1}$ extends to $\mathbb{C}$

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$H(Z+1) = H(Z) + 1 \rightsquigarrow$ horn map h map on $\mathbb{C}/\mathbb{Z}$

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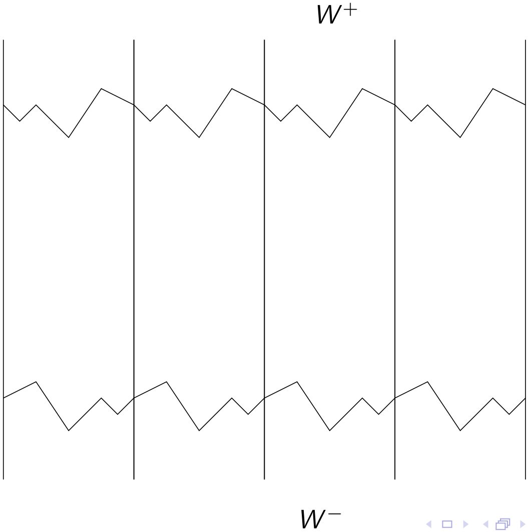

Parabolic implosion in dimension 1

Theorem (Lavaurs)

Let $f_\epsilon(z) = z + z^2 + \epsilon^2 + \dots$. Let $\epsilon_n \to 0$ s.t. $\frac{\pi}{\epsilon_n} - n \to \sigma \in \mathbb{C}$. Then $f_{\epsilon_n}^n \to \mathcal{L}_\sigma$.

Horn map $h \leftrightsquigarrow$ Lavaurs map $\mathcal{L} \simeq f_{\epsilon_n}^n$

Theorem (Lavaurs, Epstein)

If f is polynomial, then h is a finite type map.

Shishikura: $\dim_H \partial M = 2$
+parabolic/near parabolic renormalization...

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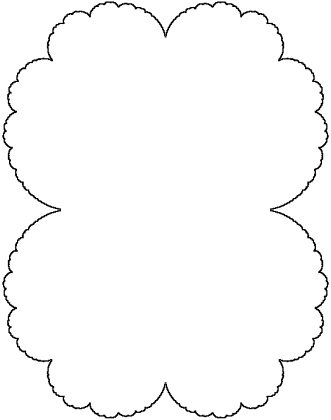
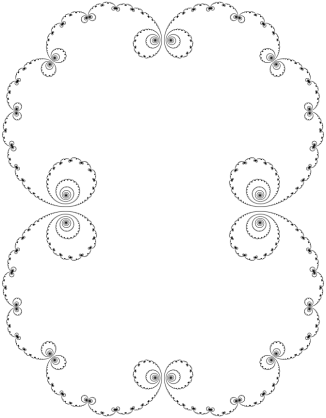
Parabolic implosion: enrichment of Julia set





Hénon maps

Hénon map: $F(x, y) = (p(x) - ay, x)$, degree $d \geq 2$
Jacobian $\mathrm{Jac}\, F(x, y) \equiv a$
If $|a| < 1$: dissipative Hénon map

- Filled-in Julia sets: $K^{\pm}(F) = \{p \in \mathbb{C}^2 : F^{\pm n}(p) \text{ bounded }\}$
- Forward and backward Julia sets: $J^{\pm} := \partial K^{\pm}$
- Green functions: $G^{\pm}(p) := \lim_{n \to +\infty} d^{-n} \log^{+} \|F^n(p)\|$
- Green currents $T^{\pm} := dd^c G^{\pm}$
- Equilibrium measure: $\mu := T^{+} \wedge T^{-}$

Bedford, Lyubich, Smillie: μ is ergodic, has entropy $\log d$, and $\chi_{\mu}^{-} < 0 < \chi_{\mu}^{+}$.

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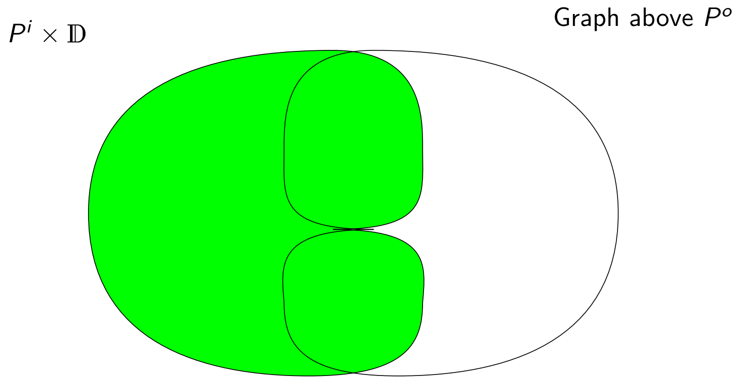
Semi-parabolic fixed point: local theory

$F(x, y) = (x + x^2 + \ldots, ay + \ldots)$ Hénon



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Ueda proved:

- $\mathcal{B}$ is open in $\mathbb{C}^2$ (and $\simeq \mathbb{C}^2$)
- there exists a submersion $\phi^\iota : \mathcal{B} \to \mathbb{C}$ s.t. $\phi^\iota \circ F = \phi^\iota + 1$. Fibers are "strongly stable manifolds" (exponential contraction)
- there exists an injective hol. map $\psi^o : \mathbb{C} \to \mathbb{C}^2$ s.t. $\Sigma := \psi^o(\mathbb{C}) = \{p \in \mathbb{C}^2 : F^{-n}(p) \to O\} \setminus O$

Definition

The Lavaurs map is $\mathcal{L} := \psi^o \circ \phi^\iota : \mathcal{B} \to \Sigma$.
The lifted horn map is $H := \phi^\iota \circ \psi^o : (\psi^o)^{-1}(\mathcal{B}) \to \mathbb{C}$. It commutes with $Z \mapsto Z + 1$.
The horn map in the induced map on $\mathbb{C}/\mathbb{Z}$.

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Parabolic implosion for semi-parabolic Hénon maps

Theorem (Bedford-Smillie-Ueda)

Let F be Hénon, $F_\epsilon(x, y) = (x + x^2 + \epsilon^2 + \ldots, ay + \ldots)$, and $|a| < 1$. Let $\epsilon_n \to 0$ such that

$$\frac{\pi}{\epsilon_n} - n \to \sigma \in \mathbb{C}.$$

Then $F^n_{\epsilon_n} \to \mathcal{L}_\sigma$ (Lavaurs map) loc. uniformly on $\mathcal{B}$.

Application: discontinuity of $J^\pm(F_\epsilon)$, $\operatorname{supp} \mu_{F_\epsilon}$, etc. at $\epsilon = 0$.

Idea: there exists σ such that $\mathcal{L}_\sigma$ has one fixed point.

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Small island property

Definition

A map $f : W \to \mathbb{P}^1$ has the small island property if for every $z_0 \in \mathbb{C}^*$, there exists $r > 0$ such that for all $U \cap \partial W$, there exists $\Omega \Subset W$ s.t.

$$f : \Omega \xrightarrow{\simeq} \mathbb{D}(z_0, r).$$

Ahlfors 5-island theorem: if $f : \mathbb{C} \to \mathbb{C}$ is meromorphic transc., given $D_1, \ldots, D_5$ disjoint Jordan domains, there exists $\Omega \Subset \mathbb{C}$ and i s.t. $f : \Omega \xrightarrow{\simeq} D_i$.

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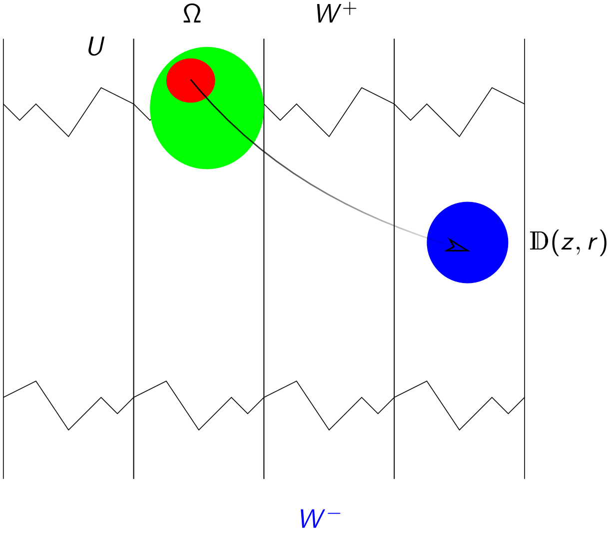

Applications

Corollary 1

Repelling cycles are dense in the Julia set of h.

Corollary 2

Assume $|\mathrm{Jac} F| < \frac{1}{d^2}$. Then there exists $F_{\epsilon_n} \to F$ s.t.

$$\dim_H J^+(F_{\epsilon_n}) \to 4.$$

Dujardin-Lyubich: if $|\mathrm{Jac} F| < \frac{1}{d^2}$, then h has a critical point.

Idea: repelling Cantor set of dimension ≈ 2 for some h_σ (Shishikura) $\rightsquigarrow$ saddle hyperbolic sets of dim. ≈ 4 for F_{ϵ_n}

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Idea of the proof of the theorem

Fact

Let H be the lifted horn map, h the horn map (on $\mathbb{C}/\mathbb{Z}$).

1. $H(w) = z \Leftrightarrow W^{ss}(p) = \{\phi^\iota = z\}$ intersects Σ at $\psi^o(w)$
2. $H'(w) \neq 0 \Leftrightarrow$ the intersection between $W^{ss}(p)$ and Σ is transverse.
3. For h: can replace $W^{ss}(p)$ by $F^{-n}(W^{ss}(p))$ and $\psi^o(w)$ by $F^n(\psi^o(w))$.

Let $D := \psi^o(U) \subset \Sigma$, W^{ss} some strong stable manifold. We want transverse intersections between $F^{-n_1}(W^{ss})$ and $F^{n_2}(D)$.

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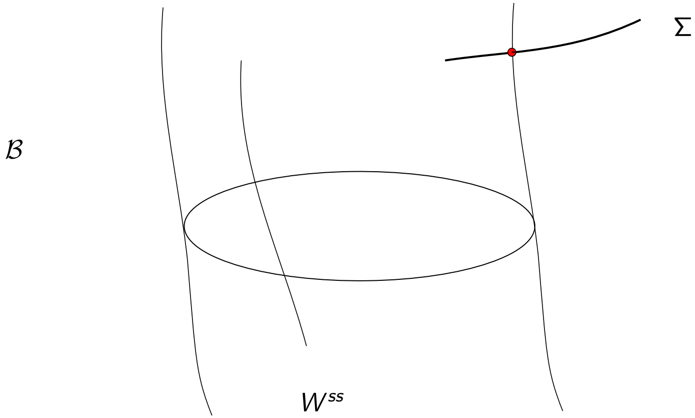

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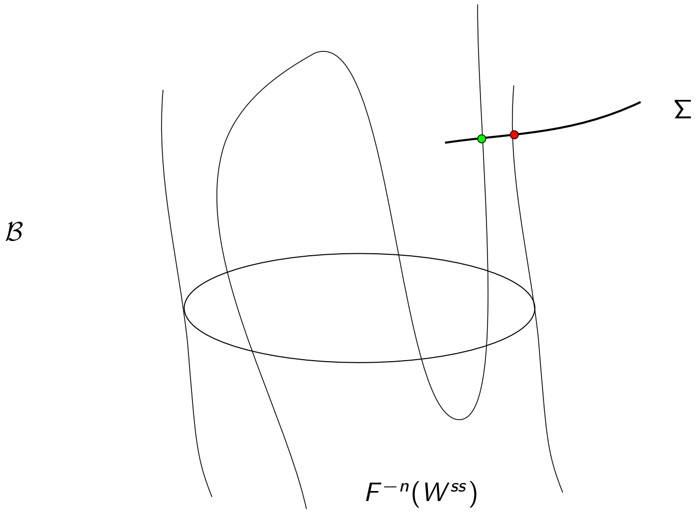

Equdistribution

Recall $T^{\pm} = dd^c G^{\pm}$, $\mathrm{supp}\, T^{\pm} = J^{\pm}$
Bedford-Smillie: if $D \cap J^{+} \neq \emptyset$, then $d^{-n}[F^n(D)] \to T^{-}$.

Let $R >> 1$, and $C :=$ connected component of $W^{ss}(p) \cap \mathbb{D}(0, R)^2$.
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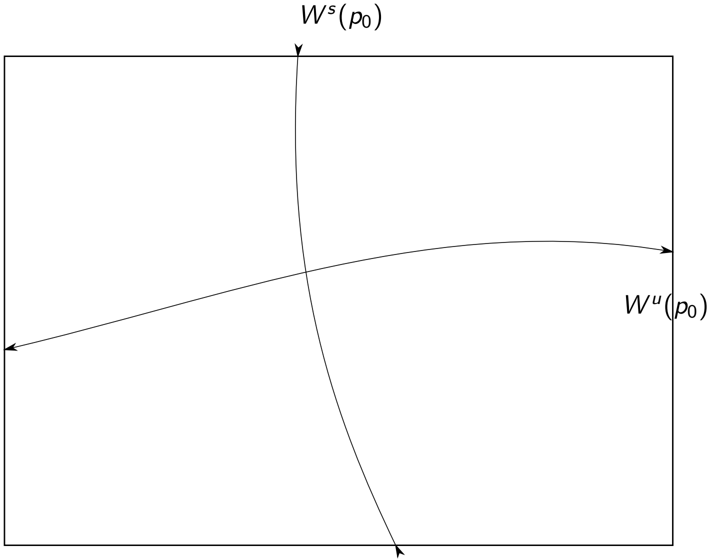

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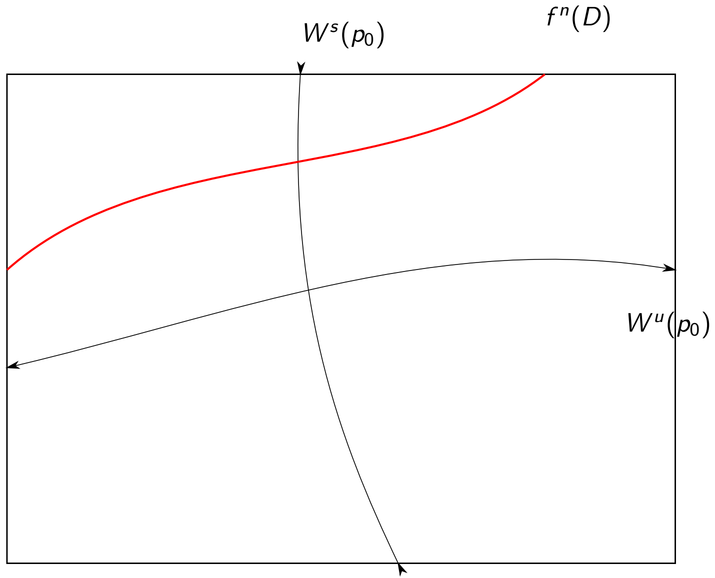

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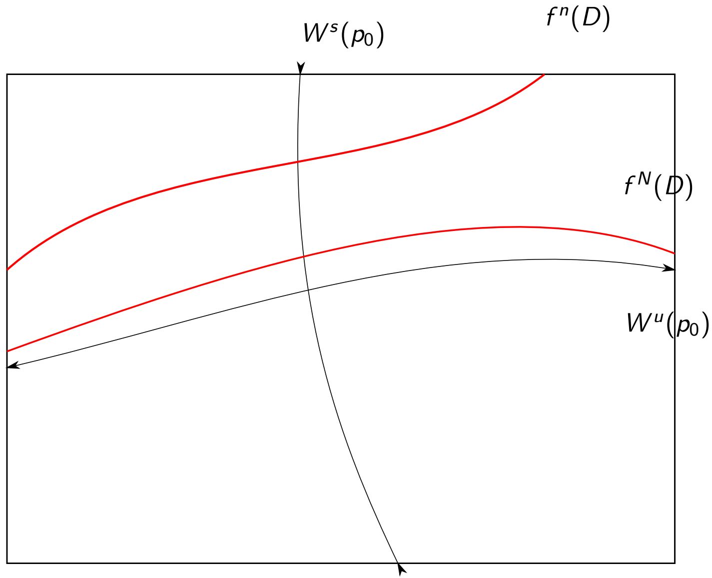

The end

Thank you for your attention