Algebraic directed immersions

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Directed immersions

- Let Y be a connected compact complex submanifold of $\mathbb{CP}^{n-1}$, $n \geq 2$.
- Consider the punctured complex cone

$$A = A(Y) = \big\{(z_1, \ldots, z_n) \in \mathbb{C}^n_* : [z_1 : \cdots : z_n] \in Y\big\}.$$

- A is smooth and connected, and its closure $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ is algebraic, and is the common zero set of finitely many homogeneous holomorphic polynomials, by Chow's theorem.

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Definition

If M is an open Riemann surface, a holomorphic immersion $M \to \mathbb{C}^n$ is said to be **directed** by A, or to be an ***A*-immersion**, if its complex derivative with respect to any local holomorphic coordinate on M takes its values in A.

- Directed immersions have been studied in many classical geometries: symplectic, contact, totally real, Lagrangian, etc.

Directed immersions

- The cone $A = A(Y)$ defines a holomorphic subbundle $\mathcal{A}$ of the vector bundle $(T^*M)^{\oplus n}$ with fibre isomorphic to A whose **sections** are vectorial $(1,0)$-forms $\alpha = (\alpha_1, \dots, \alpha_n)$ on M with no common zeros such that the ratio
$$[\alpha_1 : \cdots : \alpha_n] : M \to \mathbb{CP}^{n-1}$$
takes values in Y.

- A holomorphic map $F : M \to \mathbb{C}^n$ is an A-immersion iff dF is a holomorphic section of the bundle $\mathcal{A} \to M$.

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- If a holomorphic 1-form α is a section of $\mathcal{A} \to M$ which is **exact**,
$$\int_C \alpha = 0 \quad \text{for every closed curve } C \subset M,$$
then it determines an A-immersion $F : M \to \mathbb{C}^n$ by integration:
$$F(p) = F(p_0) + \int_{p_0}^{p} \alpha, \quad p \in M.$$

Directed immersions

- We denote by

$$\mathcal{O}^1(M, A)$$

the space of vectorial **holomorphic** 1-forms $\alpha = (\alpha_1, \ldots, \alpha_n)$ on M with no zeros which are **sections** of $\mathcal{A} \to M$.

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- Since TM and T^*M are holomorphically trivial by the Oka-Grauert principle,

there exists a holomorphic 1-form θ on M with no zeros.

Thus, every $\alpha \in \mathcal{O}^1(M, A)$ is of the form

$$\alpha = f\theta \quad \text{where} \quad f = \frac{\alpha}{\theta} \in \mathcal{O}(M, A).$$

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- This gives a natural method for constructing A-immersions $M \to \mathbb{C}^n$:

  Look for holomorphic maps $f : M \to A$ such that $f\theta$ is **exact** and integrate.

Null curves

- A cone of fundamental interest in the theory of **minimal surfaces** (locally area minimizing surfaces) in $\mathbb{R}^n$ is the (punctured) **null quadric**

$$\mathbf{A} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n_* \colon z_1^2 + \cdots + z_n^2 = 0\}, \quad n \geq 3.$$

- Holomorphic immersions directed by **A** are called **null curves**. So, a null curve $M \to \mathbb{C}^n$ is a holomorphic immersion $F = (F_1, \dots, F_n) : M \to \mathbb{C}^n$ such that $dF = (dF_1, \dots, dF_n)$ is a holomorphic section of $\mathcal{A} \to M$:

$$dF \neq 0, \quad (dF_1)^2 + \cdots + (dF_n)^2 = 0.$$

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- A holomorphic section $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathcal{O}^1(M, \mathbf{A})$,

$$\alpha \neq 0, \quad \alpha_1^2 + \cdots + \alpha_n^2 = 0,$$

  determines a null curve $F : M \to \mathbb{C}^n$

$$F(p) = F(p_0) + \int_{p_0}^{p} \alpha, \quad p \in M,$$

  iff α is exact:

$$\int_C \alpha = 0 \quad \text{for every closed curve } C \subset M.$$

Minimal surfaces

- The real and imaginary parts of a holomorphic null curve $M \to \mathbb{C}^n$ are **conformal minimal immersions** $M \to \mathbb{R}^n$; i.e., harmonic immersions $u = (u_1, \ldots, u_n) : M \to \mathbb{R}^n$ such that $\partial u \in \mathcal{O}^1(M, \mathbf{A})$:
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- A holomorphic section $\alpha \in \mathcal{O}^1(M, \mathbf{A})$ determines a conformal minimal immersion $u : M \to \mathbb{R}^n$,
$$u(p) = u(p_0) + \Re \int_{p_0}^{p} \alpha, \quad p \in M,$$
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- A holomorphic section $\alpha \in \mathcal{O}^1(M, \mathbf{A})$ determines a conformal minimal immersion $u : M \to \mathbb{R}^n$,

$$u(p) = u(p_0) + \Re \int_{p_0}^{p} \alpha, \quad p \in M,$$

  iff $\Re(\alpha)$ is exact: $\Re \int_C \alpha = 0$ for every closed curve $C \subset M$.
- The **flux map** of u is the group homomorphism $\mathrm{Flux}_u : H_1(M, \mathbb{Z}) \to \mathbb{R}^n$,

$$\mathrm{Flux}_u(C) = -\mathrm{i} \int_C \partial u = \Im \int_C \partial u, \quad [C] \in H_1(M, \mathbb{Z}).$$

  We may view Flux_u as an element of $H^1(M, \mathbb{R}^n)$.
- A conformal minimal immersion $M \to \mathbb{R}^n$ is (globally) the real part of a holomorphic null curve $M \to \mathbb{C}^n$ if and only if $\mathrm{Flux}_u = 0$.

Minimal surfaces

- The study of minimal surfaces is one of the most classical topics in mathematics. The global theory is main focus of interest since the 1960s.
- Its connection with complex analysis (Weierstrass representation) has strongly influenced the theory.
- Conformal minimal immersions $M \to \mathbb{R}^n$ admit an approximation theory which is analogous to the one of holomorphic functions on open Riemann surfaces.

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- Its connection with complex analysis (Weierstrass representation) has strongly influenced the theory.
- Conformal minimal immersions $M \to \mathbb{R}^n$ admit an approximation theory which is analogous to the one of holomorphic functions on open Riemann surfaces.
- A-López 2012 If $K \subset M$ is compact and $\mathcal{O}(M)$-convex, then every conformal minimal immersion $K \to \mathbb{R}^3$ **can be approximated** uniformly on K by **proper** conformal minimal immersions $M \to \mathbb{R}^3$.
- Mergelyan approximation is also possible.
- Tools in the proof:
  Classical Runge's theorem + fairly technical results from the function theory of Riemann surfaces + special geometry of $\mathbf{A}$ + control on the periods.

Directed immersions

- Key observation: the null quadric $\mathbf{A}$ is an **Oka manifold**, and hence there are many holomorphic maps $M \to \mathbf{A}$ and a good approximation theory.

- A-Forstnerič 2014 Systematic investigation of A-immersions.
  - Generalization of known results for minimal surfaces or null curves.
  - New results for minimal surfaces.

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- Let $A \subset \mathbb{C}^n_*$ $(n \geq 3)$ be a cone as above which is an **Oka manifold** not contained in a hyperplane. Let M be an open Riemann surface.

(i) Runge + general position: If $K \subset M$ is compact and $\mathcal{O}(M)$-convex, then every A-immersion $K \to \mathbb{C}^n$ can be approximated uniformly on K by A-**embeddings** $M \to \mathbb{C}^n$.

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(ii) Oka principle: Every continuous map $f : M \to A$ is homotopic to a holomorphic map $\tilde{f} \in \mathcal{O}(M, A)$ such that $\tilde{f}\theta$ is exact, hence $\tilde{f}\theta = d\tilde{F}$ for an A-immersion $\tilde{F} : M \to \mathbb{C}^n$.

Approximation on K where f is holomorphic.

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(iii) Under natural assumptions on A, there is a **proper** A-embedding $M \to \mathbb{C}^n$.

- (ii)-(iii) were new for minimal surfaces in $\mathbb{R}^n$ (with embeddings for $n \geq 5$).

Sketch of proof of Runge's theorem for A-immersions (I)

- Let $K \Subset L$ be a pair of $\mathcal{O}(M)$-convex compact sets in M and assume that $F : K \to \mathbb{C}^n$ is an A-immersion. Assume for simplicity that K is a strong deformation retract of L and that F is nondegenerate.

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- Let θ be a holomorphic 1-form vanishing nowhere on M, hence $dF = f\theta$ with $f \in \mathcal{O}(K, A)$. Let $\{C_1, \ldots, C_l\}$ be a basis of $H_1(K, \mathbb{Z}) \cong \mathbb{Z}^l$ such that $C = \bigcup_{j=1}^{l} C_j$ is $\mathcal{O}(M)$-convex, and define the **period map**

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathcal{C}(C, \mathbb{C}^n) \to (\mathbb{C}^n)^l, \quad \mathcal{P}_j(h) = \int_{C_j} h\theta \in \mathbb{C}^n.$$

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- Construct a **period-dominating spray** with core f: there are a neighbourhood U of 0 in $\mathbb{C}^N$ and a holomorphic map

$$\Phi_f : U \times K \to A$$

such that $\Phi_f(0, \cdot) = f$ and the map

$$U \ni \zeta \mapsto \mathcal{P}(\Phi_f(\zeta, \cdot)) \in (\mathbb{C}^n)^l$$

is **submersive** at $\zeta = 0$. Furthermore, there is a neighbourhood of f in $\mathcal{O}(K, A)$ such that the map $g \mapsto \Phi_g$ depends holomorphically on g.

Sketch of proof of Runge's theorem for A-immersions (II)

- Since A is **Oka**, we may approximate f uniformly on K by a holomorphic function $\tilde{f} : L \to A$, and $\Phi_{\tilde{f}}$ by a holomorphic period-dominating spray

$$\widetilde{\Phi}_{\tilde{f}} : U \times L \to A.$$

- If the approximation is close enough, then the image of the map $U \ni \zeta \mapsto \mathcal{P}(\widetilde{\Phi}_{\tilde{f}}(\zeta, \cdot)) \in (\mathbb{C}^n)^l$ still covers a neighbourhood of $0 \in (\mathbb{C}^n)^l$, and so there is $\zeta_0 \in U$ close to 0 such that

$$\mathcal{P}(\widetilde{\Phi}_{\tilde{f}}(\zeta_0, \cdot)) = 0;$$

  i.e, $\widehat{f} = \widetilde{\Phi}_{\tilde{f}}(\zeta_0, \cdot) : L \to A$ is close to f on K and $\widehat{f}\theta$ is **exact** on K, so on L.

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- Fixed $p_0 \in \mathring{K}$, the map $\widehat{F} : L \to \mathbb{C}^n$ given by

$$\widehat{F}(p) = F(p_0) + \int_{p_0}^{p} \widehat{f}\theta$$

  is an A-immersion and is close to F uniformly on K.

We have passed from K to L

Sketch of proof of Runge's theorem for A-immersions (III)

- Let $K = K_0 \Subset K_1 \Subset \cdots \subset \bigcup_{j\in\mathbb{N}} K_j = M$ be an exhaustion of M by $\mathcal{O}(M)$-convex compact domains.

- Set $F_0 = F$ and construct a sequence of A-immersions $F_j : K_j \to \mathbb{C}^n$ with $F_j \approx F_{j-1}$ on K_{j-1}. The limit map

$$\lim_{j\to\infty} F_j : M \to \mathbb{C}^n$$

is an A-immersion which is close to F uniformly on K. $\square$

More on directed immersions - Interpolation

- A-Castro Infantes 2019 Jet **interpolation** on a closed discrete subset of M.

More on directed immersions - Homotopies

- Forstnerič-Lárusson 2019 For a holomorphic 1-form θ with no zeros on an open Riemann surface M, the maps in (with compact-open topology)

$$\begin{array}{ccccc}
\Re\mathrm{NC}_{\mathrm{nf}}(M,\mathbb{C}^n) & \hookrightarrow & \mathrm{CMI}_{\mathrm{nf}}(M,\mathbb{R}^n) & & \\
& \searrow^{\partial} & \downarrow{\scriptstyle \partial} & & \\
& & \mathcal{O}(M,\mathbf{A}) & \hookrightarrow & \mathcal{C}(M,\mathbf{A})
\end{array}$$

where ∂ sends $u \in \mathrm{CMI}_{\mathrm{nf}}(M,\mathbb{R}^n)$ to $\partial u/\theta$, are **weak homotopy equivalences**.

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- Forstnerič-Lárusson 2019 If $A \subset \mathbb{C}^n_*$ $(n \geq 3)$ is a smooth connected cone as above, Oka and not contained in a hyperplane, then the map

$$\begin{array}{rcl} \{F : M \to \mathbb{C}^n \ : \ \ F \text{ is a nondeg. } A\text{-immersion}\} & \to & \mathcal{O}(M,A) \\ F & \mapsto & dF/\theta \end{array}$$

  is a **weak homotopy equivalence**.

  The inclusion

$$\mathcal{O}(M,A) \hookrightarrow \mathcal{C}(M,A)$$

  is also a **weak homotopy equivalence**.

CMIs and holomorphic A-immersions admit nice theories of approximation, interpolation, and homotopy.

Are there **algebraic** analogues of these results?

FTC vs Algebraic

- The natural counterpart of a meromorphic function in the theory of minimal surfaces is a **complete minimal surface of finite total curvature**.

- Chern-Osserman 1967 If $u : M \to \mathbb{R}^n$ is a complete conformal minimal immersion of FTC,

$$\int_M |K| \, d\sigma = -\int_M K \, d\sigma < +\infty,$$

then M is an **affine Riemann surface**,

$$M = \overline{M} \setminus E$$

with a compact Riemann surface $\overline{M}$ and a finite set $E = \{p_1, \ldots, p_m\} \subset \overline{M}$, $E \neq \varnothing$, and $\partial u \in \mathcal{O}^1(M, \mathbf{A})$ is **algebraic** (regular) on M with **effective poles** at all ends $p_1, \ldots, p_m$ of M.

- Complete minimal surfaces of FTC are among the most intensively studied minimal surfaces. They play an important role in the classical global theory of minimal surfaces since the seminal works by Osserman in the 1960s. They are of finite topology, are proper in space, and have a fairly simple and well-understood asymptotic behaviour.

Runge's theorem for complete minimal surfaces of FTC

- López 2014, A-López 2022 If M is an affine Riemann surface and $K \subset M$ is compact and $\mathcal{O}(M)$-convex, then every conformal minimal immersion $K \to \mathbb{R}^n$ $(n \geq 3)$ can be **approximated** uniformly on K by complete conformal minimal immersions of **finite total curvature**.

- Jet interpolation is possible on a finite set in K.

- The proof follows the scheme described above based on the control of periods. But we cannot induct, so it must be done in one shot.

- The proof relies on the classical theory of approximation and interpolation for meromorphic functions on compact Riemann surfaces (Behnke-Stein 1949, Royden 1967) and uses in a very strong way the particular geometry of the null quadric $\mathbf{A}$.

- The proof is technically very involved, requires of approxmation by meromorphic functions with control on divisors, Riemann-Roch, etc.

Algebraic directed immersions

Theorem (A-Lárusson 2023)

Let $A \subset \mathbb{C}^n_*$ $(n \geq 2)$ *be the punctured cone on a connected submanifold* Y *of* $\mathbb{CP}^{n-1}$ *and assume that* A *is* **algebraically elliptic** *and not contained in a hyperplane in* $\mathbb{C}^n$*.*

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Then the following are equivalent.

- (i) *F can be approximated uniformly on K by* **algebraic** *A-immersions $M \to \mathbb{C}^n$.*
- (ii) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|U$ that contains dF also contains the restriction of an* **algebraic** *section of $\mathcal{A}$ on M.*

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Furthermore, if (i) and (ii) hold then the algebraic A-immersion $M \to \mathbb{C}^n$ in (i) can be chosen to be **proper**.

Algebraic directed immersions

Theorem (A-Lárusson 2023)

Let $A \subset \mathbb{C}^n_*$ $(n \geq 2)$ be the punctured cone on a connected submanifold Y of $\mathbb{CP}^{n-1}$ and assume that A is **algebraically elliptic** *and not contained in a hyperplane in $\mathbb{C}^n$. Let $M = \overline{M} \setminus \{p_1, \dots, p_m\}$ be an* **affine Riemann surface** *and $\mathcal{A}$ be the subbundle of $(T^*M)^{\oplus n}$ defined by A. Let $K \subset M$ be a compact $\mathcal{O}(M)$-convex subset and $F : K \to \mathbb{C}^n$ be a holomorphic A-immersion.*
Then the following are equivalent.

(i) *F can be approximated uniformly on K by* **algebraic** *A-immersions $M \to \mathbb{C}^n$.*

(ii) *There is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|U$ that contains dF also contains the restriction of an* **algebraic** *section of $\mathcal{A}$ on M.*

Furthermore, if (i) and (ii) hold then the algebraic A-immersion $M \to \mathbb{C}^n$ in (i) can be chosen to be **proper**.

- Algebraic ellipticity is a natural condition on A which ensures that there are many algebraic maps $M \to A$ and a good approximation theory.
- Arzhantsev-Kaliman-Zaidenber 2023 Y uniformly rational $\Rightarrow$ A alg. elliptic.
- $A = \mathbf{A}$ satisfies the hypotheses and in this case (ii) always holds.

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- Interpolation is possible on a finite set in K.
- An algebraic A-immersion is proper iff it has poles at all ends $p_1, \ldots, p_m$ of M.
- (i) $\Rightarrow$ (ii) is easy.

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- Algebraic Oka theory is more rigid than standard Oka theory.
- In general, an affine Riemann surface does not carry any nowhere-vanishing algebraic 1-form: we need to work directly with sections.
- Ensuring existence of poles at all ends requires a new idea. We cannot induct.

Sketch of proof of (ii) ⇒ (i) (I)

- Let $F : K \to \mathbb{C}^n$ be a holomorphic A-immersion and assume there is a neighbourhood U of K such that the homotopy class of continuous sections of $\mathcal{A}|U$ that contains dF also contains the restriction of an algebraic section s_0 of $\mathcal{A}$ on M.

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- We can assume that K is a strong deformation retract of M (hence connected) and that F extends to a nondegenerate holomorphic A-immersion $M \to \mathbb{C}^n$ such that dF is homotopic to s_0 on M.

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- Fix a holomorphic 1-form θ vanishing nowhere on M, and set

$$f = dF/\theta : M \to A \quad \text{and} \quad f_0 = s_0/\theta : M \to A.$$

- $f, f_0 \in \mathcal{O}(M, A)$ are homotopic on M.

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- $f, f_0 \in \mathcal{O}(M, A)$ are homotopic on M.
- Let $\{C_1, \ldots, C_l\}$ be a basis of $H_1(K, \mathbb{Z}) = H_1(M, \mathbb{Z}) \cong \mathbb{Z}^l$ such that $C = \bigcup_{j=1}^{l} C_j$ is $\mathcal{O}(M)$-convex, and define the period map

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathcal{C}(C, \mathbb{C}^n) \to (\mathbb{C}^n)^l, \quad \mathcal{P}_j(h) = \int_{C_j} h\theta \in \mathbb{C}^n.$$

- Since $f\theta = dF$ is exact, we have that $\mathcal{P}(f) = 0 \in (\mathbb{C}^n)^l$.

Sketch of proof of (ii) $\Rightarrow$ (i) (II)

- Find a ball V centred at $0 \in \mathbb{C}^N$ and a holomorphic spray

$$\phi : V \times M \to A$$

with core $\phi(0, \cdot) = f$ that is period dominating:

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} \mathcal{P}(\phi(\zeta, \cdot)) : \mathbb{C}^N \to (\mathbb{C}^n)^l \quad \text{is an epimorphism.}$$

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- We may assume that ϕ extends continuously to a map $\phi : \mathbb{C}^N \times M \to A$.
- Since A satisfies the basic Oka property with approximation and interpolation, we may assume that $\phi : \mathbb{C}^N \times M \to A$ is holomorphic.

Sketch of proof of (ii) $\Rightarrow$ (i) (III)

- Using θ, we turn ϕ into a holomorphic spray $\phi\theta$ of sections of $\mathcal{A}$ on M parametrised by $\mathbb{C}^N$ such that $\phi(0, \cdot)\theta = f\theta = dF$.

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- If the approximation is close enough, then near $0 \in \mathbb{C}^N$ there is a parameter $\zeta_0 \in \mathbb{C}^n$ such that

$$\mathcal{P}(\tau(\zeta_0)/\theta) = \mathcal{P}(\phi(0,\cdot)) = \mathcal{P}(f) = 0,$$

  hence $t := \tau(\zeta_0)$ is an **exact** algebraic section of $\mathcal{A}$ on M which is close to $f\theta = dF$ on K.

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- For $p_0 \in \mathring{K}$, the algebraic A-immersion $\widetilde{F} : M \to \mathbb{C}^n$ given by

$$\widetilde{F}(p) = F(p_0) + \int_{p_0}^{p} t, \quad p \in M$$

is close to F on K, hence it satisfies (i). □

Granting poles at the ends (I)

- Let $M = \overline{M} \setminus \{p_1, \ldots, p_m\}$ be an affine Riemann surface and $F = (F_1, \ldots, F_n) : M \to \mathbb{C}^n$ be an algebraic A-immersion.

  We want to show that F may be approximated uniformly on compact sets in M by proper algebraic A-immersions $M \to \mathbb{C}^n$.

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- Let $K \subset M$ be a smoothly bounded compact domain such that

  $$\overline{M} \setminus \mathring{K} = D_1 \cup \cdots \cup D_m,$$

  where $D_1, \ldots, D_m$ are mutually disjoint closed discs centred at $p_1, \ldots, p_m$.

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- Fix points $q_i \in \mathring{D}_i \setminus \{p_i\}$, $i = 1, \dots, m$, and let

$$\widehat{F} = (\widehat{F}_1, \dots, \widehat{F}_n) : \widehat{K} = K \cup \{q_1, \dots, q_m\} \to \mathbb{C}^n$$

  be a holomorphic A-immersion such that

$$\widehat{F} = F \text{ on } K \quad \text{and} \quad |\widehat{F}_1(q_i)| > \max\{|F_1(p)| : p \in bD_i\} \text{ for } i = 1, \dots, m.$$

Granting poles at the ends (II)

- Since F is algebraic so is dF, and hence condition (ii) holds for $d\widehat{F}$ on a neighbourhood of $\widehat{K}$, and hence (i) implies that we can approximate $\widehat{F}$ on $\widehat{K}$, hence F on K, by an algebraic A-immersion $\widetilde{F} : M \to \mathbb{C}^n$.

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This implies that $\widetilde{F}_1 : M \to \mathbb{C}$ is a proper map.

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- $\widetilde{F} : M \to \mathbb{C}^n$ is a **proper** algebraic A-immersion which is close to F on K. $\square$

Homotopies of complete minimal surfaces of FTC

Theorem (A-Forstnerič-Lárusson 2024)

If M is an affine Riemann surface, then the maps in the diagram

$$\begin{array}{ccc} \Re\mathrm{NC}_*(M,\mathbb{C}^n) & \hookrightarrow & \mathrm{CMI}_*(M,\mathbb{R}^n) \\ & \searrow^{\partial} & \downarrow \partial \\ & & \mathscr{A}^1(M,\mathbf{A}) \end{array}$$

where $\mathscr{A}^1(M,\mathbf{A}) \subset \mathcal{O}^1(M,\mathbf{A})$ *is the subspace of* **algebraic** *sections of* $\mathcal{A} \to M$ *and* ∂ *is the* $(1,0)$*-differential, are weak homotopy equivalences.*
($_*$ *=***complete, nonflat, and FTC***.)*

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- The inclusions $\mathscr{A}^1_\infty(M,\mathbf{A}) \hookrightarrow \mathscr{A}^1_*(M,\mathbf{A}) \hookrightarrow \mathscr{A}^1(M,\mathbf{A})$, where $_\infty$=**nonflat with poles at the ends**, and $_*$=**nonflat**, are weak homotopy equivalences.

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- The proof relies on algebraic Oka theory.

  It requires of an improved version of the algebraic homotopy approximation theorem for sections of algebraically elliptic submersions over an affine manifold, due to Forstnerič.

Theorem (A-Forstnerič-Lárusson 2024)

Let M be an affine Riemann surface, and let $A \subset \mathbb{C}^n_*$, $n \geq 2$, be a **flexible** *smooth connected cone as above that is not contained in any hyperplane. Then the map*

$$\mathscr{I}_*(M, A) \to \mathscr{A}^1(M, A), \quad h \mapsto dh$$

from the space $\mathscr{I}_*(M, A)$ of nondegenerate proper **algebraic** *A-immersions $M \to \mathbb{C}^n$ to the space $\mathscr{A}^1(M, A)$ of* **algebraic** *1-forms on M with values in A is a weak homotopy equivalence.*

- For an algebraic manifold, being flexible is a stronger condition than being algebraically elliptic.

The Mittag-Leffler theorem

- Mittag-Leffler 1884 If $E \subset \mathbb{C}$ is closed and discrete and f is meromorphic on a neighborhood of E then there is a meromorphic function $\tilde{f}$ in $\mathbb{C}$ such that $\tilde{f}$ is holomorphic on $\mathbb{C} \setminus E$ and $\tilde{f} - f$ is holomorphic at all points in E.

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- A-López 2022 Let M be an open Riemann surface, $E \subset M$ be a closed discrete subset, $r : E \to \mathbb{N}$ be a map, and u be a conformal minimal immersion from a punctured neighbourhood of E into $\mathbb{R}^n$ $(n \geq 3)$ such that every point in E is a **complete end of finite total curvature** of u.

  Then there is a **complete** conformal minimal immersion $\tilde{u} : M \setminus E \to \mathbb{R}^n$ of **finite total curvature** such that $\tilde{u} - u$ is **harmonic** and vanishes to order $r(p)$ at every point $p \in E$.

Theorem (A-Vrhovnik, work in progress)

Let $A \subset \mathbb{C}^n_*$ $(n \geq 3)$ *be a cone as above which is an* **Oka manifold** *not contained in a hyperplane. Let* M *be an open Riemann surface,* $E \subset M$ *be a closed discrete subset,* $r : E \to \mathbb{N}$ *be a map, and* F *be a holomorphic* A*-immersion from a punctured neighbourhood of* E *into* $\mathbb{C}^n$ *such that* F *has an* **effective pole** *at each point of* E*.*

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*Then, for any map $r : E \to \mathbb{N}$ there is a meromorphic map $\widetilde{F} : M \to \mathbb{C}^n$ such that $\widetilde{F}$ is a holomorphic A-***embedding** *on $M \setminus E$ such that $\widetilde{F} - F$ is* **holomorphic** *and vanishes to order $r(p)$ at every point $p \in E$.*

The Mittag-Leffler theorem for directed immersions

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Furthermore, under natural assumptions on the cone A, the holomorphic A-immersion $\widetilde{F} : M \setminus E \to \mathbb{C}^n$ can be chosen to be **proper***, hence a proper embedding.*

Algebraic directed immersions

Antonio Alarcón

Joint work with Finnur Lárusson

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