Laplacians on (infinite) metric graphs

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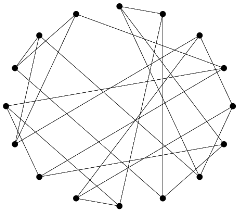
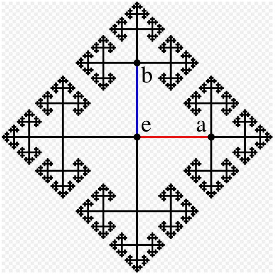
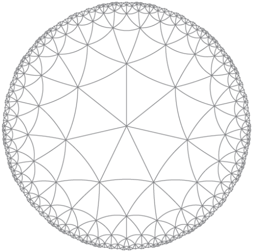
Graphs

Definition

A **graph** $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a set of **vertices** $\mathcal{V}$ and **edges** $\mathcal{E}$.







Assumptions

- $\mathcal{V}$ and $\mathcal{E}$ are at most countable, and $\mathcal{G}_d$ is **connected**
- $\mathcal{G}_d$ is **locally finite** (vertex degree: $\deg(v) < \infty$, $v \in \mathcal{V}$)

Metric graphs

Definition (a.k.a. “cable graphs” or “metrized graphs”)

$\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ is a connected, locally finite graph.
If every edge $e \in \mathcal{E}$ is assigned with a positive finite length $|e| \in (0, \infty)$, then $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$ is called a metric graph

Metric Graph as ...

- a simplicial 1-complex,

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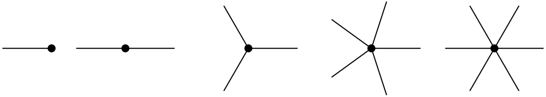
Metric Graph as ...

- a simplicial 1-complex,
- a topological space, which looks locally like a star-graph



- a length space when equipped with a natural ("geodesic") path metric – a distance between two points is the arc-length of "shortest" path,
- a (real) 1D manifold with singularities: vertices of degree ≥ 3 are "branching" points; degree $= 1$ are "boundary" points,
- a non-Archimedean analog of Riemann surfaces

  a *tropical curve* or a degeneration of a smooth family of Riemann surfaces

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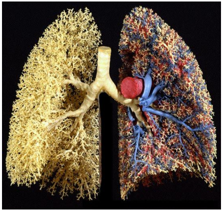
Applications: "thin wire materials" in physics/biology/...



lungs $\approx$ binary tree of 20-23 generations
approx. $2 \times 10^6 - 1.6 \times 10^7$ *vertices*

Cast of human lungs (photo by E. Weibel)

P. Joly, M. Kachanovska, and A. Semin, *Netw. Heterog. Media* (2019)

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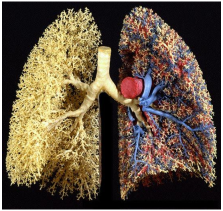

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G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, AMS, 2013

P. Exner and H. Kovařík, *Quantum waveguides*, Springer, 2015

(Kirchhoff) Laplacians on metric graphs

Given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, |\cdot|)$, identify each edge $e \in \mathcal{E}$ with $\mathcal{I}_e = [0, |e|]$. Let

$$L^2(\mathcal{G}) \cong \bigoplus_{e\in\mathcal{E}} L^2(\mathcal{I}_e).$$

Kirchhoff Laplacian ("Laplace–Beltrami" on $\mathcal{G}$)

Δ acts as $-\frac{\mathrm{d}^2}{\mathrm{dx}_e^2}$ on the interior of $\mathcal{G}$, and boundary conditions:

$$\text{Kirchhoff conditions:} \quad \begin{cases} f \text{ is continuous at } v \\ \sum_{e\in\mathcal{E}_v} \partial_e f(v) = 0 \end{cases}, \quad v \in \mathcal{V}.$$

- $\deg(v) = 1$: Kirchhoff = Neumann at v, $\partial_e f(v) = 0$,
- $\deg(v) = 2$: Kirchhoff = continuity of f and its derivative at v ("removable" singularity/inessential vertex)

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The **maximal Kirchhoff Laplacian** Δ_{Kir} is defined in $L^2(\mathcal{G};\mu)$ on the domain

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \big\{ f \in H^2(\mathcal{G}\setminus\mathcal{V}) |\ (\text{Kirchhoff}) \text{ on } \mathcal{V} \big\}.$$

The **minimal Kirchhoff Laplacian** $\Delta_{\mathrm{Kir},0}$ is the L^2 closure of

$$\Delta \restriction \mathrm{dom}(\Delta_{\mathrm{Kir}}) \cap L^2_c(\mathcal{G}).$$

Self-adjointness (a.k.a. Quantum Completeness)

Problem: Do we need a boundary condition at "infinity"?

When $\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$, i.e., when Δ_{Kir} is self-adjoint?

NOTE: Since $\Delta_{\mathrm{Kir},0} \geq 0$,

$$\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}} \Leftrightarrow L^2\text{-}\textbf{uniqueness for Schrödinger and Wave equations}$$

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§VIII.11 Three mathematical problems in quantum mechanics

(1) *Self-adjointness*: ... The first mathematical problem is to prove essential self-adjointness or, if the operator is not essentially self-adjoint, to investigate the various self-adjoint extensions and choose the "right one" to be the observable.

M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I: Functional Analysis*, Academic Press, 1980.

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von Neumann formulas

$$\mathrm{dom}(\Delta_{\mathrm{Kir}}) = \mathrm{dom}(\Delta_D) \dot{+} \ker(\Delta_{\mathrm{Kir}} - \lambda), \quad \lambda \in \mathbb{C} \setminus \sigma(\Delta_D).$$

Δ_D is the Dirichlet Laplacian (the Friedrichs extension of $\Delta_{\mathrm{Kir},0}$)

Since $\ker(\Delta_{\mathrm{Kir}} - \lambda) = L^2$ λ-harmonic functions and $\sigma(\Delta_D) \subseteq [0, \infty)$:

- self-adjoint uniqueness $\Leftrightarrow$ no L^2 harmonic f-ns (λ-harmonic with $\lambda < 0$),
- description of self-adjoint extns = description of L^2 λ-harmonic functions!

Graph Boundaries

Poisson = bounded harmonic; Martin = positive harmonic, ...

Self-adjointness (a.k.a. Quantum Completeness)

For manifolds: Cauchy boundary $\partial_C M = \overline{M} \setminus M$; completeness is $\partial_C M = \emptyset$
completeness $\Rightarrow$ self-adjoint uniqueness (Gaffney'54; Roelcke'60; Chernoff'73).

Gaffney-type Theorem on Metric graphs

$\Delta_{\mathrm{Kir},0} = \Delta_{\mathrm{Kir}}$ if $(\mathcal{G}, \varrho_{\mathrm{intr}})$ is complete.

The proof is based on Yau's L^p-Liouville theorem for strongly local Dirichlet forms:

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$$\mathrm{dom}(\Delta_G) = \{f \in \mathrm{dom}(\Delta_{\mathrm{Kir}}) \,|\, \operatorname{grad} f \in L^2\} = \mathrm{dom}(\Delta_{\mathrm{Kir}}) \cap H^1(\mathcal{G}).$$

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Definition

Markovian uniqueness: $H^1(\mathcal{G}) = H^1_0(\mathcal{G})$.

- Does $\mathrm{e}^{-\Delta t}$ give rise to a unique positivity preserving and L^∞ contractive semigroup?

Boundaries for infinite graphs: Graph Ends

Definition (Freudenthal, 1944; Halin, 1964):

A **ray** in $\mathcal{G}$ is an infinite path $\mathcal{R} = (v_n)_{n\geq 0}$ without self-intersections.
Two rays are **equivalent**, if they **cannot be separated** by cutting out a finite set of vertices

A graph end is an equivalence class of rays. The set of ends is $\mathfrak{C}(\mathcal{G})$.

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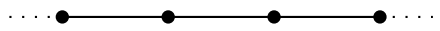

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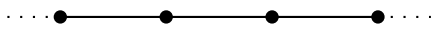

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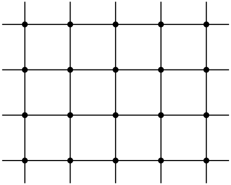

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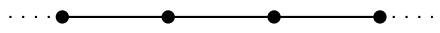

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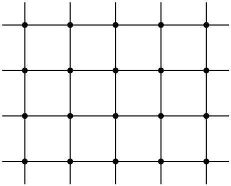
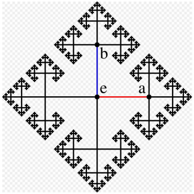


∞ many ends



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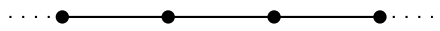

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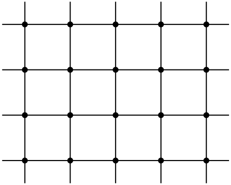

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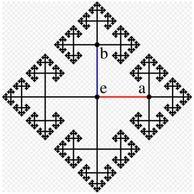


Theorem (Hopf–Freudenthal, 1940's; Stallings, 1968):

If $\mathcal{G}$ is a **Cayley graph** of a finitely generated group, then

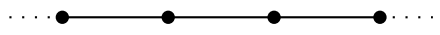
$$\#\mathfrak{C}(\mathcal{G}) \in \{0, 1, 2, \infty\}.$$

Finite Volume graph ends

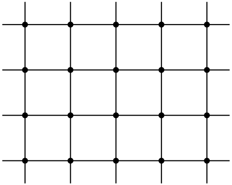
Definition

- A graph end γ has finite volume, if γ has a **neighborhood** U (w.r.t. the Freudenthal compactification) such that $\mathrm{vol}(U) < \infty$.
- The set of finite volume ends is denoted by $\mathfrak{C}_0(\mathcal{G})$.

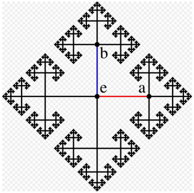
For the following examples, a graph end has finite volume, if...



... its ray has **finite length**



... the graph has **finite total volume**



... it has a **"subtree"** of finite total volume

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Graph ends and $H^1(\mathcal{G})$

$\mathfrak{C}_0(\mathcal{G})$ is a **proper boundary** for H^1-functions!

- Every $f \in H^1(\mathcal{G})$ has a **continuous extension** to $\mathcal{G} \cup \mathfrak{C}_0(\mathcal{G})$.
- General framework: ideal boundaries by **Gelfand theory**

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Theorem (AK–Mugnolo–Nicolussi' 2022)

$$H_0^1(\mathcal{G}) = \{f \in H^1(\mathcal{G}) |\ f(\gamma) = 0,\ \gamma \in \mathfrak{C}_0(\mathcal{G})\}.$$

In particular, $H_0^1(\mathcal{G}) = H^1(\mathcal{G}) \iff \mathfrak{C}_0(\mathcal{G}) = \emptyset$.

Graph Ends and Markovian extensions

Corollary (AK–Mugnolo–Nicolussi' 2022)

The following are equivalent:

- $H_0^1(\mathcal{G}) = H^1(\mathcal{G})$,
- $\Delta_{\mathrm{Kir},0}$ has a **unique Markovian extension**,
- The **Gaffney Laplacian** Δ_G is **self-adjoint**,
- All graph ends have **infinite volume**, $\mathfrak{C}_0(\mathcal{G}) = \emptyset$.

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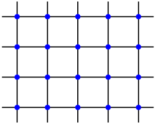
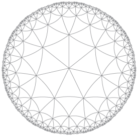
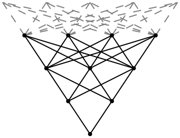
- The **Gaffney Laplacian** Δ_G is **self-adjoint**,
- All graph ends have **infinite volume**, $\mathfrak{C}_0(\mathcal{G}) = \emptyset$.

- If $\mathcal{G}$ has only one graph end, $\#\mathfrak{C} = 1$:

$\Delta_{\mathrm{Kir},0}$ has a **unique Markovian extension** $\iff \mathrm{vol}(\mathcal{G}) = \sum_{e\in\mathcal{E}} |e| = \infty$

Amenable groups (Γ is not amenable if a Ponzi scheme exists on Γ)

Theorem (Stallings)

If a finitely generated group Γ is amenable, then $\#\mathfrak{C}(\Gamma) < \infty$.

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If Γ is amenable$^{(*)}$, then $H^1(\mathcal{G}) = H^1_0(\mathcal{G})$ for any $\mathcal{G}$ with $\mathrm{vol}(\mathcal{G}) = \infty$.

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Theorem (AK–Nicolussi' 2019)

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Understand groups via random walks:

- Tarski monster (A.Yu. Olshanskii'1980); free Burnside group (S.I. Adyan'1982);
- Basilica group (Bartholdi&Virag'2005)

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Γ is not amenable $\Leftrightarrow$ $\Delta_{\mathrm{Kir},0} > 0$ for any $\mathcal{G}$ with $\sup_{e\in\mathcal{E}} |e| < \infty$.

H. Kesten, *Symmetric random walks on groups*, TAMS (1958).

Understand groups via random walks:

- Tarski monster (A.Yu. Olshanskii'1980); free Burnside group (S.I. Adyan'1982);
- Basilica group (Bartholdi&Virag'2005)

Problem: Is the **Thompson group** amenable?

Amenable groups (Γ is not amenable if a Ponzi scheme exists on Γ)

Theorem (Stallings)

If a finitely generated group Γ is amenable, then $\#\mathfrak{C}(\Gamma) < \infty$.

Corollary

If Γ is amenable$^{(*)}$, then $H^1(\mathcal{G}) = H^1_0(\mathcal{G})$ for any $\mathcal{G}$ with $\mathrm{vol}(\mathcal{G}) = \infty$.

Theorem (AK–Nicolussi' 2019)

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Understand groups via Brownian motion? Thank you for your attention!