A Geometric approach to Polynomial and Rational Approximation

Kirill Lazebnik, based on joint work with Chris Bishop.

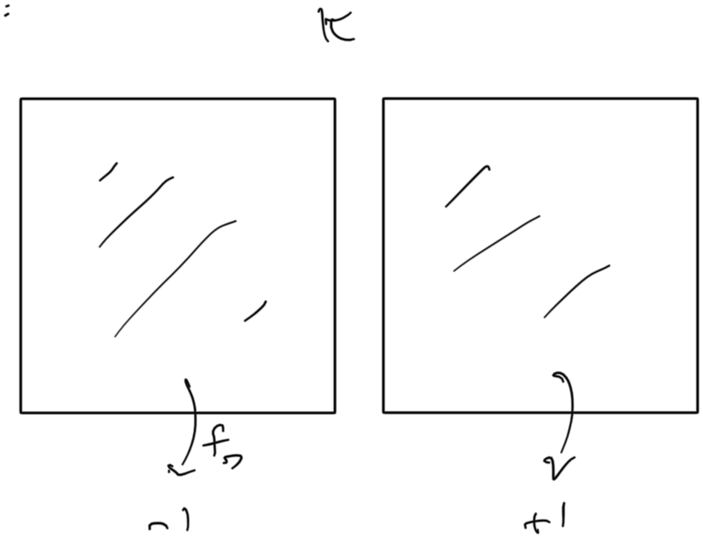
Thm: (Runge) Suppose $K \subseteq \mathbb{C}$ cpt, $\mathbb{C} \setminus K$ connected, f holomorphic in a nbhd. of K, $\varepsilon > 0$. Then $\exists$ polyn. p s.t.

$$\sup_{z \in K} |p(z) - f(z)| < \varepsilon.$$

Goal: construct approximants p with good understanding of p off K.

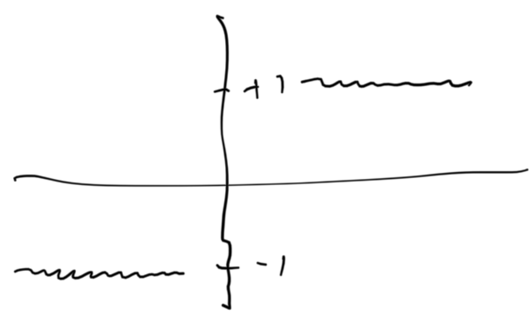
this is not trivial?

e.g.:



Runge $\Rightarrow$

$|p_n - f_n| < 1/n$



Our approach to "Runge" is as follows

① approximate $f: K \to f(K)$ by a proper holomorphic

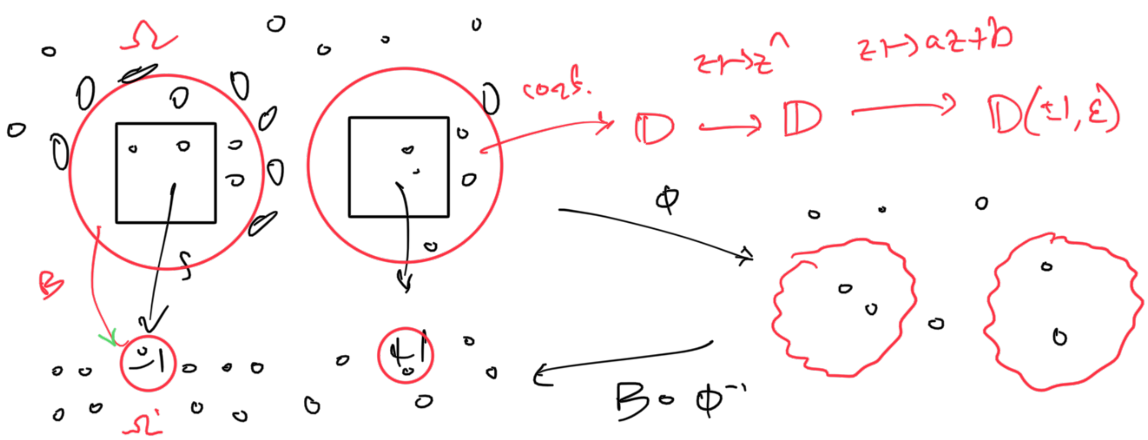
$$B: \Omega \supseteq K \to \Omega' \supseteq f(K) \quad (\Omega, \Omega' \text{ open})$$

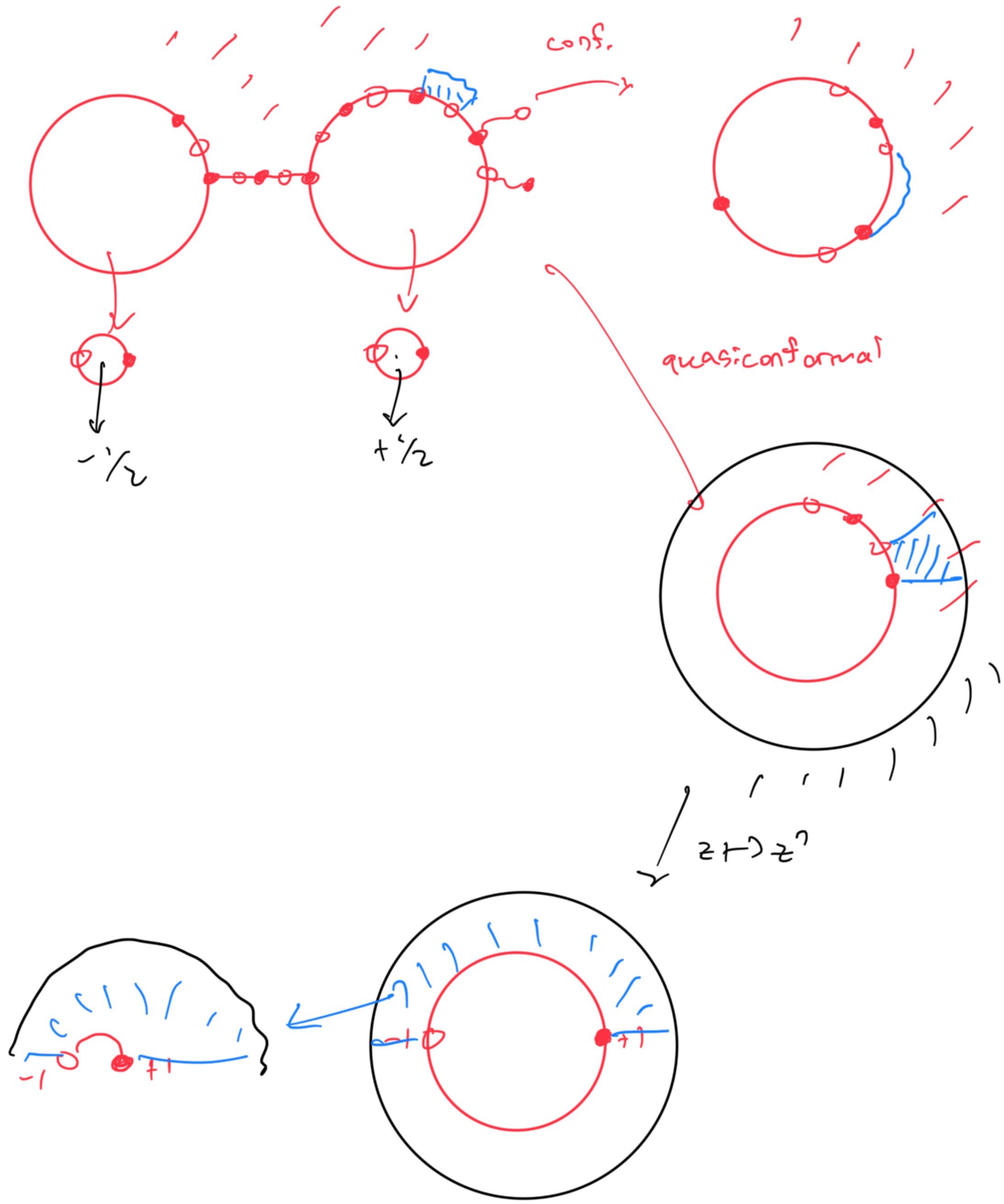
② extend $B: \Omega \to \Omega'$ quasiregularly to $B: \mathbb{C} \to \mathbb{C}$, such that B is "close" to holomorphic i.e. has "small dilatation"

③ Apply Measurable Riemann Mapping Thm. to give quasiconformal $\Phi: \mathbb{C} \to \mathbb{C}$ s.t. $B \circ \Phi^{-1}$ is holom. (hence a polynomial) so that

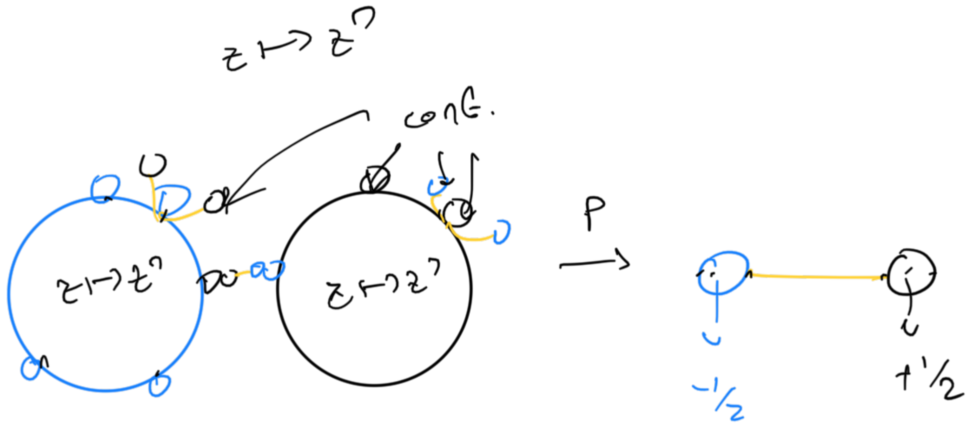
$$B \circ \Phi^{-1} \approx B \approx f \quad \text{on } K$$

$\Phi(z) \approx z$





conf.
quasiconformal
-1/2
+1/2
z ↦ z^7
-1
+1


z ↦ z^7
conj.
z ↦ z^7
z ↦ z^7
P
-1/2
+1/2