Bergman metric as a pull-back of the Fubini-Study metric

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• Let Ω be a domain in $\mathbb{C}^n$. Define the Bergman space $A^2(\Omega)$ to be the set of all holomorphic functions f on Ω such that

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• Assume that $A^2(\Omega) \neq \{0\}$. Then $A^2(\Omega)$ is a non-trivial Hilbert subspace of $L^2(\Omega)$. Let $\{\phi_j\}_{j=1}^N$ be an orthonormal basis for $A^2(\Omega)$ with N being either finite or $+\infty$.

- The Bergman kernel function of Ω is defined by

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- When $\Omega=B_n$, the unit ball in $\mathbb{C}^n$, we have

$$K_{B_n}(z,w)=\frac{1}{v(B_n)}\frac{1}{(1-\langle z,w\rangle)^{n+1}}.$$

- Let Ω_j $(j = 1, 2)$ be two domains in $\mathbb{C}^n$ and let $F : \Omega_1 \to \Omega_2$ be a biholomorphic map. Then

$$K_{\Omega_1}(z, w) = \det F'(z) K_{\Omega_2}(F(z), F(w)) \overline{\det F'(w)}.$$

When $K(z) = K(z,z) > 0$ over Ω, $\omega_\Omega := i\partial\overline{\partial}K(z) \geq 0$, called the Bergman form of Ω.

When $\omega_\Omega > 0$, it associates a well-defined Kähler metric over Ω, called the Bergman metric of Ω. The transformation formula now gives the following invariant property of the Bergman metrics:

$$F*(\omega_{\Omega_2}) = \omega_{\Omega_1}.$$

Still let $\Omega \subset \mathbb{C}^n$ be a domain. We say that
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2. $A^2(\Omega)$ separates holomorphic directions if for any non-zero $X_p \in T_p^{(1,0)}\Omega$ there is a $\phi \in A^2(\Omega)$ such that $X_p(\phi)(p)) \neq 0$.

3. We say that $A^2(\Omega)$ separates points of Ω if $A^2(\Omega)$ is base-point free and the associated *Bergman-Bochner map* F defined by

$$F(z) = [\phi_1(z), \cdots, \phi_m(z), \cdots, \cdots, \phi_N] : \Omega \to \mathbb{P}^N(\mathbb{C}).$$

is one-to-one on Ω. Here $\{\phi_j\}_{j=1}^N$ is an orthonormal basis of $A^2(\Omega)$. (N is either finite or $N = \infty$).

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- Any bounded domain is an admissible domain.

Let F be the Bergman-Bochner map (which is unique up to a linear isometric isomorphism of $\mathbb{P}^N$. Then

$$\omega_\Omega = i\partial\overline{\partial}\log(\sum_{j=1}^{N}|\phi_j(z)|^2) := F^*(\omega_{st}).$$

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The Fubini-Study metric, denoted by ω_{st}, of $\mathbb{P}^\infty$ with homogeneous coordinates $[z_1, \cdots, z_n, \cdots, \cdots]$ is formally defined by

$$\omega_{st} := i\partial\overline{\partial}\log\left(\sum_{j=1}^{\infty}|z_j|^2\right)$$

• Kobayashi (1959) If Ω is an admissible domain in $\mathbb{C}^n$, then the Bergman metric

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• (Ohsawa, 1981) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^1-smooth boundary. Then the Bergman metric ω_Ω is complete.

More classical results on the Bergman metrics:

•. (Bergman, 1942; Kobayashi, 1959) Holomorphic sectional curvature of a Bergman metric is always bounded by the one for a complex projective space.

•. (Fuks, 1966) Ricci curvature of the Bergman metric of a bounded domain is bounded by $(n+1)$.

- No example of a domain with no low bound for the Bergman holomorphic sectional curvatures was constructed or known.

•. (Klembeck, 1978) The holomorphic sectional curvature of the Bergman metric of a bounded strongly pseudo-convex domain approaches to a negative constant for the ball with the same dimension as the base point approaches to the boundary.

•. (Fu-Wong, 1997, Shafikov-Nemirovski, 2002 ≤ 2; Huang-Xiao, 2022, > 2) Any bounded strongly pseudoconvex domain whose Bergman metric has a constant Ricci-curvature is biholomorphic to the unit ball of the same dimension. Just heard a talk that the result holds for bounded finite type domains in $\mathbb{C}^2$ by Savale-Xiao.

- If (M, g) is Kähler then M has constant holomorphic sectional curvatures if and only if

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = -\frac{\kappa}{2}(g_{\alpha\overline{\beta}}g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}}g_{\gamma\overline{\beta}}),$$

where $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ is the curvature tensor given by

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \frac{\partial^2 g_{\alpha\overline{\beta}}}{\partial z_\gamma \partial \overline{z_\delta}} - \sum_{\lambda,\mu} g^{\lambda\overline{\mu}} \frac{\partial g_{\gamma\overline{\mu}}}{\partial z_\alpha} \frac{\partial g_{\lambda\overline{\delta}}}{\partial \overline{z_\beta}}.$$

• Bergman metric on the unit ball has a negative constant holomorphic sectional curvature. In fact

$$g_{\alpha\overline{\beta}}(z) = \frac{n+1}{1-|z|^2}(\delta_{\alpha\beta} + \frac{\overline{z}_\alpha z_\beta}{1-|z|^2})$$

and

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \frac{1}{n+1}\left(g_{\alpha\overline{\beta}}g_{\gamma\overline{\delta}} + g_{\alpha\overline{\delta}}g_{\gamma\overline{\beta}}\right)$$

with

$$\kappa = -\frac{2}{n+1} < 0.$$

Theorem

Let (M^n, g) be a complete Kähler manifold with constant holomorphic curvature. Then its universal covering space is analytically isometrically equivalent with

1) $(\mathbb{P}^n, \lambda\omega_{st})$ *if* $\kappa > 0$, *where* $\lambda > 0$ *is a certain positive constant;*

2) $\mathbb{C}^n$ *if* $\kappa = 0$;

3) $(\mathbb{B}_n, \lambda\omega_{Berg})$ *if* $\kappa < 0$, *where* $\lambda > 0$ *is a certain positive constant.*

Let $n > 1$ and let

$$\Omega = \{z \in \mathbb{C}^n : |z| < 1, z_n \neq 0\}.$$

Then
a) Ω is pseudoconvex;

b) the Bergman metric ω_Ω is not complete;

c) ω_Ω has a constant holomorphic sectional curvature, which is the same as that for ω_{B_n}.

Calabi's theorems

• The Fubini-Study metric, denoted by ω_{st}, of $\mathbb{P}^\infty$ with homogeneous coordinates $[z_1, \cdots, z_n, \cdots, \cdots]$ is formally defined by

$$\omega_{st} := i\partial\overline{\partial} \log \Big(\sum_{j=1}^{\infty} |z_j|^2 \Big)$$

Calabi's theorems

• Let (M, ω) be a complex manifold and let $F : (M, \omega) \to (\mathbb{P}^\infty, \omega_{st})$ be a holomorphic map. We say that F is a local holomorphic isometric embedding from M into $\mathbb{P}^\infty$ if

$$F^*(\omega_{st}) = i\partial\overline{\partial} \log \Big(\sum_{j=1}^{\infty} |f_j(z)|^2 \Big) = \omega \quad \text{over } U \subset M$$

for any local hol. representation $F = [f_1, \cdots, f_n, \cdots]$ over U.

Calabi's theorems

Observation: **Let F be a Bergman-Bochner map from $(D, \omega_D) \to \mathbb{P}^\infty$. Assume that D is Bergman admissible. Then F is a holomorphic isometric embedding.**

Calabi's theorems

The following two fundamental theorems of Calabi proved in 1953 will play a key role in our study of Bergman metrics.

Theorem

Let (M, ω) be a complex manifold. Let F and $G : (M, \omega) \to (\mathbb{P}, \omega_{st})$ be two holomorphic maps such that

$$F^*(\omega_{st}) = G^*(\omega_{st}).$$

Then there is a linear isometric isomorphism $T : X_F \to X_G$ such that $G = T \circ F$, where X_F and X_G are the smallest closed subspaces containing $F(M)$ and $G(M)$, respectively.

Calabi's theorems

Theorem

Let (M^n, ω) be a Kähler manifold of dimension n with a real analytic Kähler metric ω. Let $U \subset M$ be a neighborhood of $z^0 \in U$ and let $F : U \to \mathbb{P}^\infty$ be a local holomorphic isometric embedding. Then for any continuous curve $\gamma : [0,1] \to M$ with $\gamma(0) = z^0$, F extends holomorphically along γ as a local holomorphic isometric embedding.

Negative holomorphic sectional curvature

Using the Bergman-Bochner map as the isometric map and employing Calabi rigidity and extension theorems, we are able to prove the following uniformization theorem.

Theorem

(Huang and Li, 2023) Let Ω be an admissible pseudoconvex domain in $\mathbb{C}^n$. Then the Bergman metric ω_Ω has a negative constant holomorphic sectional curvature if and only if Ω is biholomorphic to $\mathbb{B}_n \setminus E$, where E is a pluripolar subset which is relatively closed in $\mathbb{B}_n$.

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Remark: The theorem remains true for Stein manifolds.

Negative holomorphic sectional curvature

a) When Ω is bounded and ω_Ω is complete, the above uniformization theorem is the classical result by Lu Qi-Keng in 1966, where $E = \emptyset$.

Negative holomorphic sectional curvature

a) When Ω is bounded and ω_Ω is complete, the above uniformization theorem is the classical result by Lu Qi-Keng in 1966, where $E = \emptyset$.

b) When Ω is bounded and the Bergman kernel satisfies the condition:

$$\frac{K(z,z)K(p,p)}{|K(z,p)|^2} = \infty, z \in \partial\Omega$$

for some $p \in \Omega$, the above uniformization theorem was proved by Dong and Wong in 2020.

c). Our theorem solves a folklore open question.

Negative holomorphic sectional curvature

Let Ω be an admissible pseudo-convex domain such that ω_Ω has constant holomorphic sectional curvature.

Step 1. Use Calabi rigidity theorem and Calabi holomorphic extension theorem to prove: There is a biholomorphic map $F : \Omega \to D$ with D is a subdomain in $\mathbb{B}_n$.

Negative holomorphic sectional curvature

Step 2. Also, we have $\omega_D = \lambda \omega_{B_n}$ for some $\lambda > 0$. Using the reproducing property of Bergman kernel function, we then prove that there is a non-constant holomorphic function h on B_n such that any function $f \in A^2(D)$ can be extended to a holomorphic function in $\mathcal{O}(B_n \setminus Z(h))$.

Negative holomorphic sectional curvature

Step 3. From the result in Step 2, one dimensional classical result and Ohsawa-Takegoshi theorem, one proves that $E := B_n \setminus D$ is a pluripolar set which is relatively closed in B_n.

Positive holomorphic sectional curvature

Is there an admissible domain $\Omega \subset \mathbb{C}^n$ such that its Bergman metric ω_Ω has positive constant holomorphic sectional curvature?

Theorem

Let Ω be a complex manifold with its Bergman space being infinite dimensional. Suppose that Ω is Bergman admissible. Then the Bergman metric of Ω cannot have a positive constant holomorphic sectional curvature.

Ω is Bergman admissible if $A^2(\Omega)$ is base point free, separate points and holomorphic directions.

Positive holomorphic sectional curvature

Let

$$D(\alpha) = \left\{(z, w) : \Big| |w| - |z| \Big| < \frac{1}{(1 + |z| + |w|)^{\alpha}}\right\}.$$

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Theorem

(H-Li, 2023) For any $2 < \alpha < 3$, the following statements hold:
(i) The Bergman space

$$A^2(D(\alpha)) = \mathit{Span}\{1, z, w\}.$$

(ii) The Bergman metric has constant holomorphic sectional curvature 2.

Positive holomorphic sectional curvature

Conjecture

Let M be a Bergman admissible domain. Then the Bergman metric of M cannot have a constant non-negative holomorphic sectional curvature.

Positive holomorphic sectional curvature

Observation 1: There is an old conjecture dating back to Wiegerinck in 1984 that asserts that the Bergman space of a pseudo-convex domain in $\mathbf{C}^{\mathbf{n}}$ is either trivial or of infinite dimension. (It is true in the one dimension case by the work of L. Carleson.) If this conjecture is true, then any pseudoconvex domain in $\mathbb{C}^n$ can not carry a Bergman metric with constant positive HS curvature.

Positive holomorphic sectional curvature

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Observation 2: Wiegerinck's conjecture holds for Hartogs pseudoconvex domains by the work of P. Jucha (2012).

Zero holomorphic sectional curvatures

QUESTION. Is there a domain $\Omega \subset \mathbb{C}^n$ such that its Bergman metric ω_Ω has constant zero holomorphic sectional curvature?

Zero holomorphic sectional curvatures

Theorem

(Huang-Li, 2023) Let Ω be a complex manifold with a non-constant bounded holomorphic function. Suppose that Ω is admissible. Then the Bergman metric of Ω can not have constant zero holomorphic sectional curvature.

Zero holomorphic sectional curvatures

Define the Hartogs domain D as follows:

$$D = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : |w|^2 < e^{-\|z\|^2}\}$$

Then D is unbounded pseudoconvex with a real analytic defining function

$$\rho(z, w) = 2 \log |w| + \|z\|^2$$

Let ω_D be the Bergman metric of D. Then

$$i : (\mathbb{C}^n, \omega_{\text{eucl}}) \to (D, \omega_D)$$

with $i(z) = (z, 0)$ is a totally geodesic embedding.

Zero holomorphic sectional curvatures

Open Question: *Construct an unbounded pseudoconvex domain whose Bergman metric is flat, or prove such a domain does not exist.*

Four conjectures

Conjecture 1 *Let M be a Stein manifold of complex dimension at least two. Suppose that Ω is Bergman admissible. Then the Bergman metric of M cannot have a constant non-negative holomorphic sectional curvature.*

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Conjecture 2 (Yau's open problem book) *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $n \geq 2$. If the Bergman metric ω_Ω is Einstein and complete, then Ω is biholomorphic to a bounded homogenous domain.*

Four conjectures

Conjecture 3 If the Bergman metric in the smooth part of a normal Stein space M with a smoothly compact strongly pseudoconvex boundary is Einstein, then M is non-singular and thus is biholomorphic to the ball by the work of Huang-Xiao.

Conjecture 4

The Bergman metric of a smoothly bounded pseudo-convex domain of finite D'Angelo type is pinched by two negative constants near the boundary.

Thanks

THANK YOU!