Positivity of Kähler-Einstein currents

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$\hookrightarrow$ *The Calabi conjecture* remained open for two decades [Yau 78].

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$\hookrightarrow$ Construction of Ricci flat Kähler metrics when $c_1(X) = 0$.

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Theorem (Li 22)

A $\mathbb{Q}$-Fano variety X admits a singular K-E metric iff it is (unif.) K-stable.

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Strict positivity: local models

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$\hookrightarrow$ 3-dimensional setting uses a combination of these 4 ingredients !

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Theorem (DiNezza-G-Guenancia 20)

Consider, for $t \neq 0$, $\varphi_t \in \mathcal{C}^\infty(X_t)$ the unique solution to

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  [$\alpha = \frac{1}{nd}$ OK if X_t degree d subvariety of $\mathbb{P}^N$]

Tool 2: Chern-Lu inequality

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- Observe that $\omega_t \leq C\omega_{KE,t} \Longleftrightarrow \mathrm{Tr}_{\omega_{KE,t}}(\omega_t) \leq C$.

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- Since $(\omega_{KE,t})^n/\omega_t^n$ is uniformly bounded away from X_0^{sing}, we obtain locally unif. higher order estimates in $\mathcal{X}^{reg}$ and pass to the lim on X_0.

Global smoothing, non zero curvature

Consider, for $t \neq 0$, $\lambda \neq 0$, $\varphi_t \in \mathcal{C}^\infty(X_t)$ canonical solution to

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$\hookrightarrow$ Quite a difficult estimate ! Extension to pairs by [Pan-Trusiani 23].

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- Use of pairs necessary in negative curvature (more involved than CY).

A few recent references

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