Gromov ellipticity in complex analytic geometry and algebraic geometry

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Example

A Riemann surface Y is Kobayashi hyperbolic if and only if Y is universally covered by the unit disc $\mathbb{D}$.

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  $\implies$ Demailly's algebraic hyperbolicity:
  $\exists \varepsilon > 0 \, \forall C \subset Y^{\text{proj}}$: *curve* $\quad 2\,\mathrm{genus}(C) - 2 \geq \varepsilon \deg(C)$
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Let X be a complex space and Y be a complex manifold.

1. A *spray* over a holomorphic map $f : X \to Y$ is a holomorphic map $s : E \to Y$ from a holomorphic vector bundle E over X such that $s(0_x) = f(x)$ for each $x \in X$.

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2. A family of sprays $s_\lambda : E_\lambda \to Y$ $(\lambda \in \Lambda)$ over a holomorphic map $f : X \to Y$ is *dominating* if $\sum_{\lambda \in \Lambda} (ds_\lambda)_{0_x}(T_{0_x}(E_\lambda)_x) = T_{f(x)}Y$ for each $x \in X$.

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Example

Y: complex homogeneous ($:\Longleftrightarrow \exists G \curvearrowright Y$: holomorphic transitive)
$s : Y \times T_{1_G}G \to Y$, $s(y, v) = \exp(v) \cdot y$: dominating spray $/\mathrm{id}_Y$

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Problem (open)

subelliptic $\implies$ *elliptic?*

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1. Y *is elliptic.*
2. Y *is subelliptic.*
3. Y *satisfies* Ell_1.
4. Y *is not Kobayashi hyperbolic.*
5. $Y \cong \mathbb{P}^1, \mathbb{C}, \mathbb{C}^*$ *or elliptic curve.*

Example (Lárusson '13, Lárusson–Truong '19, K. '21)

Every smooth toric variety is elliptic.

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Theorem (Oka '39, Grauert '57, '58)

For a Stein space X and $r \in \mathbb{N}$,

$$\{\textit{hol. vec. bdl. of rank } r/X\} \xrightarrow{\textit{forgetful}} \{\textit{top. vec. bdl. of rank } r/X\}$$

induces a bijection between the sets of isomorphism classes.

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$$\left\{ \tilde{f} \in \mathcal{O}(X, Y) : \tilde{f}|_Z = f \right\} \hookrightarrow \left\{ \tilde{f} \in \mathcal{C}(X, Y) : \tilde{f}|_Z = f \right\}$$

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The Oka principle for sections of *subelliptic submersions* also holds.
$\implies$ Grauert's Oka principle for isom. classes of vector bundles

Forstnerič's Oka principle and Oka manifolds

Definition

Y enjoys the *Convex Approximation Property (CAP)* if $\overline{\mathcal{O}(\mathbb{C}^n, Y)|_K} = \mathcal{O}(K, Y)$ for any compact convex $K \subset \mathbb{C}^n$ $(n \in \mathbb{N})$.

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Theorem (Forstnerič's Oka principle '09)

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Y is an *Oka manifold* if Y enjoys the above property ($\iff$ CAP).

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$\exists$ *non-subelliptic Oka manifold (e.g.* $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ *for* $n \geq 3$*)*

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$\rightsquigarrow$ Forstnerič–Wold: simpler proof ('20), $\mathbb{C}^n \setminus$(unbdd cvx set) ('23)

Surjective holomorphic maps onto Oka manifolds

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Remark

If there exists a surjective holomorphic map $\mathbb{C}^n \to Y$, for any Stein space X and 0-dimensional closed (possibly nonreduced) complex subspace $Z \subset X$ the restriction $\mathcal{O}(X, Y) \to \mathcal{O}(Z, Y)$ is surjective.

Algebraic dominating sprays and algebraic ellipticity

Algebraic dominating sprays and algebraic ellipticity conditions can be defined analogously by replacing analytic objects as follows:

- complex space $\rightsquigarrow$ algebraic variety
- holomorphic map $\rightsquigarrow$ (algebraic) morphism
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Remark

algebraic $\mathrm{Ell}_1 \implies$ unirational and
nondegenerate: $\nexists Y \to \mathbb{A}^1 \setminus \{0\}$: nonconst.

Equivalences between algebraic ellipticity conditions

Theorem (Gromov '89, Forstnerič '06, Lárusson–Truong '19, Kaliman–Zaidenberg '23)

For a smooth complex algebraic variety Y, TFAE:

1. *Y is algebraically elliptic.*

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$\rightsquigarrow$ algebraic $\mathrm{Ell}_1 \implies$ analytic Ell_1

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Example (Lárusson–Truong '19)

A smooth toric variety Y is alg. elliptic $\iff$ Y is nondegenerate.

algebraic ellipticity $\iff$ algebraic Oka property?

Theorem (Lárusson–Truong '19)

For any proper smooth complex algebraic variety Y there exist a smooth closed algebraic subvariety $Z \subset \mathbb{C}^2$ and a null-homotopic map $f \in \mathcal{O}_{alg}(Z, Y) \setminus \mathcal{O}_{alg}(\mathbb{C}^2, Y)|_Z$. In particular,

$$\left\{\tilde{f} \in \mathcal{O}_{alg}(\mathbb{C}^2, Y) : \tilde{f}|_Z = f\right\} \hookrightarrow \left\{\tilde{f} \in \mathcal{C}(\mathbb{C}^2, Y) : \tilde{f}|_Z = f\right\}$$

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Another way is to replace the interval object $[0, 1]$ by $\mathbb{A}^1$.

Surjective morphisms onto algebraically elliptic varieties

Theorem (K. '22: arXiv:2212.06412)

For any algebraically elliptic (irreducible) variety Y, there exists a morphism $f : \mathbb{A}^{\dim Y+1} \to Y$ such that $f(\mathbb{A}^{\dim Y+1} \setminus \mathrm{Sing}(f)) = Y$.

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For any affine variety X, any algebraically elliptic variety Y and any zero-dimensional closed (possibly nonreduced) subscheme $Z \subset X$ the restriction $\mathcal{O}_{alg}(X, Y) \to \mathcal{O}_{alg}(Z, Y)$ is surjective.

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Q. How can we generalize this for $Z \subset X$ of arbitrary dimension?

Conjectures

Conjecture ($\mathbb{A}^1$-Homotopy Extension Property)

Let X be an affine variety, Y be an algebraically elliptic variety, $f : X \to Y$ be a morphism, $Z \subset X$ be a closed algebraic subvariety and $H : Z \times \mathbb{A}^1 \to Y$ be a morphism such that $H(\cdot, 0) = f|_Z$.

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The analytic versions of the above conjectures hold!

Thank you for your attention!