Finite jet determination of non-collapsing holomorphic maps

Bernhard Lamel, Nordine Mir, and Guillaume Rond

Complex Analysis, Geometry, and Dynamics
Portorož 2023

Holomorphic maps between real submanifolds of complex spaces

- $M \subset \mathbb{C}^n$ generic submanifold ("source")
- $M' \subset \mathbb{C}^{n'}$ generic submanifold ("target")
- (nearly all of the time) real-analytic/real-algebraic

Holomorphic maps between real submanifolds of complex spaces

- $M \subset \mathbb{C}^n$ generic submanifold ("source")
- $M' \subset \mathbb{C}^{n'}$ generic submanifold ("target")
- (nearly all of the time) real-analytic/real-algebraic

Definition

We say that a germ of a holomorphic map $H\colon (M, p) \to \mathbb{C}^{n'}$ takes M into M' if there exists a neighbourhood U of p such that $H(M \cap U) \subset M'$.

Holomorphic maps between real submanifolds of complex spaces

- $M \subset \mathbb{C}^n$ generic submanifold ("source")
- $M' \subset \mathbb{C}^{n'}$ generic submanifold ("target")
- (nearly all of the time) real-analytic/real-algebraic

Definition

We say that a germ of a holomorphic map $H\colon (M,p) \to \mathbb{C}^{n'}$ takes M into M' if there exists a neighbourhood U of p such that $H(M \cap U) \subset M'$.

Definition

If $H = \sum_\alpha H_\alpha (z-p)^\alpha \in \mathbb{C}[[z-p]]^{n'}$ is a formal holomorphic map, we call

$$j_p^k H := \sum_{|\alpha| \leq k} H_\alpha (z-p)^\alpha$$

the k-jet of H at the point p.

Finite jet determination

Finite jet determination Problem

Given M, M', does there exist k such that for any two germs $H, \tilde{H} \colon (M, p) \to M'$, if $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$?

Finite jet determination

Finite jet determination Problem

Given M, M', does there exist k such that for any two germs $H, \tilde{H} \colon (M, p) \to M'$, if $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$?

Motivation–Cartan's theorem

If $H \colon \Omega \to \Omega$ is a holomorphic automorphism of a *bounded* domain in $\mathbb{C}^n$, and $H(p) = p$, $H'(p) = I$, (i.e. $j_p^1 H = j_p^1 \mathrm{id}$) then $H(z) = z$.

Finite jet determination

Finite jet determination Problem

Given M, M', does there exist k such that for any two germs $H, \tilde{H} \colon (M, p) \to M'$, if $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$?

Motivation–Cartan's theorem

If $H \colon \Omega \to \Omega$ is a holomorphic automorphism of a *bounded* domain in $\mathbb{C}^n$, and $H(p) = p$, $H'(p) = I$, (i.e. $j_p^1 H = j_p^1 \mathrm{id}$) then $H(z) = z$.

Finite determination "at the boundary"

Boundary values of automorphisms are *not* necessarily determined by their 1-jets any longer:

$$H_t(z_1, z_2) = \left(\frac{z_1}{1 - tz_2}, \frac{z_2}{1 - tz_2} \right) = (z_1, z_2) + \dots$$

is a 1-parameter family stabilizing $\mathrm{Im} z_2 = |z_1|^2$.

Assumptions

$$\mathbb{C}TM \supset (T^{1,0}\mathbb{C}^n \cap \mathbb{C}TM) \oplus (T^{0,1}\mathbb{C}^n \cap \mathbb{C}TM) =: T^{1,0}M \oplus T^{0,1}M$$

Definition (Minimality)

We say that M is minimal (at p) if the Lie algebra generated by sections of $T^{1,0}M \oplus T^{0,1}M$ generates $\mathbb{C}T_pM$.

Definition (Holomorphic nondegeneracy)

We say that M' is holomorphically nondegenerate if $T^{1,0}M$ does not possess any holomorphic sections.

Two equidimensional theorems

Theorem (Baouendi-Ebenfelt-Rothschild, Baouendi-Mir-Rotschild)

If $M, M' \subset \mathbb{C}^n$ are minimal holomorphically nondegenerate, then finite jet determination holds for biholomorphisms.

Two equidimensional theorems

Theorem (Baouendi-Ebenfelt-Rothschild, Baouendi-Mir-Rotschild)

If $M, M' \subset \mathbb{C}^n$ are minimal holomorphically nondegenerate, then finite jet determination holds for biholomorphisms.

Theorem (Ebenfelt-L-Zaitsev, Juhlin)

If $M, M' \subset \mathbb{C}^n$ are holomorphically nondegenerate hypersurfaces, then finite jet determination holds for biholomorphisms.

A couple of remarks

- Holomorphic nondegeneracy is necessary: Complex-time flows of holomorphic sections of $T^{1,0}M$ are not finitely determinable.
- Minimality is also *necessary* if one wants unique determination in the (wider) class of smooth CR diffeomorphisms (Kossovskiy-L, Kossovskiy-L-Stolovitch)
- We're not talking about smooth CR diffeomorphisms too much in this talk, but for these Ebenfelt (hypersurface) and Kim-Zaitsev (higher codimension and even abstract) provided finite determination in the minimal k-nondegenerate case.
- A lot of activity recently for automorphisms of (higher-codimensional) smooth Levi-nondegenerates, by Bertrand, Blanc-Centi, Meylan, Tumanov.

Higher codimension $n' > n$

Mappings between balls

Erik Løw (1985) shows that if $\mathbb{B} \subset \mathbb{C}^n$, $\mathbb{B} \subset \mathbb{C}^{n'}$ and $n' >> n$, then unique finite jet determination at an interior point is impossible for proper maps of *low boundary regularity*.

Higher codimension $n' > n$

Mappings between balls

Erik Løw (1985) shows that if $\mathbb{B} \subset \mathbb{C}^n$, $\mathbb{B} \subset \mathbb{C}^{n'}$ and $n' >> n$, then unique finite jet determination at an interior point is impossible for proper maps of *low boundary regularity*.

Mappings between spheres

If we are looking at proper mappings between balls which have *smooth* boundary values, then Forstnerič (1989) shows that

- these maps are rational;
- and of bounded degree.

So unique jet determination is possible in that case.

Recent positive results: Target spheres (Mir-Zaitsev) and strictly pseudoconvex real-algebraic targets (L-Mir)

Enemies of finite determination

Assume from now on that M is minimal, and that $M' \subset \mathbb{C}^{n'}$ is a Nash set. If M' contains a holomorphic curve, all (general) hope is lost:

Example

Assume that $A\colon \Delta \to \mathbb{C}^{n'}$ is a holomorphic curve, $A(\Delta) \subset M'$. Then *any* holomorphic map $h\colon (\mathbb{C}^n, p) \to (\mathbb{C}, 0)$ gives rise to a holomorphic map $A \circ h\colon (M, p) \to M'$. Such maps cannot be determined by a jet of finite order.

Enemies of finite determination

Assume from now on that M is minimal, and that $M' \subset \mathbb{C}^{n'}$ is a Nash set. If M' contains a holomorphic curve, all (general) hope is lost:

Example

Assume that $A\colon \Delta \to \mathbb{C}^{n'}$ is a holomorphic curve, $A(\Delta) \subset M'$. Then *any* holomorphic map $h\colon (\mathbb{C}^n, p) \to (\mathbb{C}, 0)$ gives rise to a holomorphic map $A \circ h\colon (M, p) \to M'$. Such maps cannot be determined by a jet of finite order.

Definition

We say that $q \in M'$ is of *infinite type* if there exists a complex variety X, with $q \in X \subset M'$. We denote the set of infinite type points in M' by $\mathcal{E}_{M'}$.

Enemies of finite determination

Assume from now on that M is minimal, and that $M' \subset \mathbb{C}^{n'}$ is a Nash set. If M' contains a holomorphic curve, all (general) hope is lost:

Example

Assume that $A\colon \Delta \to \mathbb{C}^{n'}$ is a holomorphic curve, $A(\Delta) \subset M'$. Then *any* holomorphic map $h\colon (\mathbb{C}^n, p) \to (\mathbb{C}, 0)$ gives rise to a holomorphic map $A \circ h\colon (M, p) \to M'$. Such maps cannot be determined by a jet of finite order.

Definition

We say that $q \in M'$ is of *infinite type* if there exists a complex variety X, with $q \in X \subset M'$. We denote the set of infinite type points in M' by $\mathcal{E}_{M'}$.

Definition

$H\colon (M, p) \to M'$ is *non-collapsing* if $H(M) \not\subset \mathcal{E}_{M'}$.

Positive Results

Theorem (L-Mir-Rond 2023)

Let $M \subset \mathbb{C}^n$ be a minimal real-analytic CR submanifold and $M' \subset \mathbb{C}^{n'}$ a Nash set. Then for any compact C there exists a k such that for any non-collapsing maps $H, \tilde{H} \colon (M, p) \to M'$, with $p \in C$, if $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$.

Positive Results

Theorem (L-Mir-Rond 2023)

Let $M \subset \mathbb{C}^n$ be a minimal real-analytic CR submanifold and $M' \subset \mathbb{C}^{n'}$ a Nash set. Then for any compact C there exists a k such that for any non-collapsing maps $H, \tilde{H} \colon (M, p) \to M'$, with $p \in C$, if $j_p^k H = j_p^k \tilde{H}$, then $H = \tilde{H}$.

As a consequence, we get:

Corollary (L-Mir-Rond 2023)

Let $M \subset \mathbb{C}^n$ be a minimal real-analytic CR submanifold and $M' \subset \mathbb{C}^{n'}$ a Nash set containing no complex-analytic subvariety of positive dimension. Then for any compact $C \subset M$ there exists a k such that for any two holomorphic maps $H, \tilde{H} \colon (M, p) \to M'$, if $j_p^k H = j_p^k \tilde{H}$, with $p \in C$, then $H = \tilde{H}$.

Boundary values of proper maps

Recall that we know some boundary regularity is *necessary* for finite determination.

Corollary

Let $\Omega \subset \mathbb{C}^n$ *and* $\Omega' \subset \mathbb{C}^{n'}$ *be bounded domains with, respectively, smooth real-analytic boundary and smooth real-algebraic boundary. Then there exists an integer* ℓ*, depending only on* $\partial\Omega$ *and* $\partial\Omega'$*, such that if* $F, G \colon \Omega \to \Omega'$ *are two proper holomorphic mappings extending smoothly up to the boundary near some point* $p \in \partial\Omega$ *with* $j_p^\ell F = j_p^\ell G$*, it follows that* $F = G$*.*

The (strong) finite jet determination property

Definition

A subsheaf $\mathcal{S}$ of the sheaf of holomorphic maps from M into M' is said to have the (strong) finite jet determination property if for every $p \in M$ there exists an integer k_p (uniformly bounded on compact subsets of M) such that if $j_p^{k_p} H = j_p^{k_p} \tilde{H}$, then $H = \tilde{H}$.

So now we are going to prove the strong finite jet determination for the sheaf $\mathcal{S}$ of non-collapsing maps.

Theorem (A)

With M, M' fixed as above the following holds. There exists an integer ℓ, depending only on ρ, and integer r_ℓ, a universal polynomial map $\mathcal{P}$ and a collection $\mathbb{S}$ of finitely many (universal) rational maps such that for every $p \in M$ and $f \in \mathcal{S}_p$, there exist $\mathcal{Q} \in \mathbb{S}$ such that f satisfies the following set of equations near p:

$$\begin{cases} \mathcal{P}\left(A, \overline{A}, \left(\rho_{\alpha\bar{\beta}}(f,\bar{f})\right)_{|\alpha|+|\beta|\leq\ell}\right) = 0, \qquad \bar{L}A = 0, \\ A = \mathcal{Q}\left(\left(j^k(f,\bar{f})\right)_{k\leq r_\ell}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\leq r_\ell}\right) \end{cases}$$

Furthermore, for such a choice of $\mathcal{Q}$, the system is non-degenerate in the sense that its generic rank e_f (defined in next page) is full ($= n'$).

The generic rank e_f of the previous system for each $f \in \mathcal{S}_p$ is **the generic rank with respect to f of its prolongation with respect to $\bar{L}$**, i.e. the that of the matrix formed with n' columns with

$$\bar{L}^{\delta}\partial_w \left(\mathcal{P}\left(\mathcal{Q}\left(\left(j^k(f,\bar{f})\right),(\rho_{\nu\bar{\gamma}}(w,\bar{f}))\right),\overline{A},\left(\rho_{\alpha\bar{\beta}}(w,\bar{f})\right)\right)\right)\Big|_{w=f},$$

$$\bar{L}^{\delta}\bar{L}\,\partial_w\left(\mathcal{Q}\left(\left(j^k(f,\bar{f})\right),(\rho_{\nu\bar{\gamma}}(w,\bar{f}))\right)\right)\Big|_{w=f},\quad |\delta|\le n',\ \delta\in\mathbb{N}^m.$$

The generic rank e_f of the previous system for each $f \in \mathcal{S}_p$ is **the generic rank with respect to** f **of its prolongation with respect to** $\bar{L}$, i.e. the that of the matrix formed with n' columns with

$$\bar{L}^{\delta}\partial_w \left(\mathcal{P}\left(\mathcal{Q}\left(\left(j^k(f,\bar{f})\right),(\rho_{\nu\bar{\gamma}}(w,\bar{f}))\right),\overline{A},\left(\rho_{\alpha\bar{\beta}}(w,\bar{f})\right)\right)\right)\Big|_{w=f},$$

$$\bar{L}^{\delta}\bar{L}\,\partial_w\left(\mathcal{Q}\left(\left(j^k(f,\bar{f})\right),(\rho_{\nu\bar{\gamma}}(w,\bar{f}))\right)\right)\Big|_{w=f}, \quad |\delta| \le n',\ \delta \in \mathbb{N}^m.$$

Theorem (B)

If M and M' are as above and if $\mathcal{S}$ is any subsheaf of $\mathcal{F}(M,M')$ satisfying a universal non-degenerate system as above, then there exists $K \in \mathbb{Z}_+$ such the K-jet mapping $j_p^K|_{\mathcal{S}_p}$ is injective for every $p \in M$.

- Theorem (B) is proved building on earlier work (L-Mir 2022).

- Theorem (B) is proved building on earlier work (L-Mir 2022).
- We shall focus on a sketch of proving Theorem (A)

- Theorem (B) is proved building on earlier work (L-Mir 2022).
- We shall focus on a sketch of proving Theorem (A)
- We introduce a process called *refined CR prolongation*, linking the geometry of the maps with the CR geometry of M'.

- Theorem (B) is proved building on earlier work (L-Mir 2022).
- We shall focus on a sketch of proving Theorem (A)
- We introduce a process called *refined CR prolongation*, linking the geometry of the maps with the CR geometry of M'.
- The construction can be applied to any map f (not necessarily non-collapsing). However, Theorem (A) only holds for the sheaf of non-collapsing maps $\mathcal{S}$.

Let M, M' be as above. Fix an integer ℓ and suppose that there is a universal polynomial map $\mathcal{P}$, and a collection $\mathbb{S}^\ell$ of finitely many (universal) rational maps such that for every $p \in M$ and $f \in \mathcal{F}_p$, there exist $\mathcal{Q} \in \mathbb{S}^\ell$ such that f satisfies near p

$$\begin{cases} \mathcal{P}\left(A, \overline{A}, \left(\rho_{\alpha\bar\beta}(f,\bar f)\right)_{|\alpha|+|\beta|\le\ell}\right) = 0, \qquad \bar L A = 0, \\ A = \mathcal{Q}\left(\left(j^k(f,\bar f)\right)_{k\le r_\ell}, \left(\rho_{\nu\bar\gamma}(f,\bar f)\right)_{|\nu|+|\gamma|\le r_\ell}\right) \end{cases}$$

Then for every $f \in \mathcal{F}(M, M')$, let e_f be the generic rank of the matrix formed with n' columns with

$$\bar L^\delta \partial_w \left(\mathcal{P}\left(\mathcal{Q}\left(\left(j^k(f,\bar f)\right), \left(\rho_{\nu\bar\gamma}(w,\bar f)\right)\right), \overline{A}, \left(\rho_{\alpha\bar\beta}(w,\bar f)\right)\right)\right)\Big|_{w=f},$$

$$\bar L^\delta \bar L\, \partial_w \left(\mathcal{Q}\left(\left(j^k(f,\bar f)\right), \left(\rho_{\nu\bar\gamma}(w,\bar f)\right)\right)\right)\Big|_{w=f}, \quad |\delta| \le n',\ \delta \in \mathbb{N}^m,$$

taken as the maximum value among all possible choices of $\mathcal{Q}$'s and set

$$\kappa_f := n' - e_f.$$

- κ_f is the degeneracy of f (with respect to the given systems).

- κ_f is the degeneracy of f (with respect to the given systems).
- If for every $p \in M$ and every $f \in \mathcal{F}_p(M, M')$, $\kappa_f = 0$, there is nothing to be done.

- κ_f is the degeneracy of f (with respect to the given systems).
- If for every $p \in M$ and every $f \in \mathcal{F}_p(M, M')$, $\kappa_f = 0$, there is nothing to be done.
- Assume that there exists one $g \in \mathcal{F}_p(M, M')$ for some $p \in M$ such that $\kappa_g > 0$.

Starting with

$$\begin{cases} \mathcal{P}\left(A, \overline{A}, \left(\rho_{\alpha\bar{\beta}}(f,\bar{f})\right)_{|\alpha|+|\beta|\le \ell}\right) = 0, \qquad \bar{L}A = 0, \\ A = \mathcal{Q}\left(\left(j^k(f,\bar{f})\right)_{k\le r_\ell}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\le r_\ell}\right) \end{cases}$$

the refined CR prolongation gives another universal system

$$\begin{cases} \mathcal{P}^{\#}\left(A^{\#}, \overline{A^{\#}}, \left(\rho_{\alpha\bar{\beta}}(f,\bar{f})\right)_{|\alpha|+|\beta|\le \ell+1}\right) = 0, \qquad \bar{L}A^{\#} = 0, \\ A^{\#} = \mathcal{Q}^{\#}\left(\left(j^k(f,\bar{f})\right)_{k\le r_{\ell+1}}, (\rho_{\nu\bar{\gamma}}(f,\bar{f}))_{|\nu|+|\gamma|\le r_{\ell+1}}\right), \end{cases}$$

and such that

$$\kappa_f^{\#} \le \kappa_f.$$

We apply the above construction in an iterative way and start with the system

$$\rho(f, \bar{f}) = 0$$

with associated degeneracy κ_f^1, and inductively build the ℓ-th order refined CR prolongation of the initial system and associated degeneracies $\kappa^{\ell+1}$. Hence for every $p \in M$ and every $f \in \mathcal{F}_p(M, M')$, we have the decreasing sequence of degeneracies

$$\kappa_f^1 \geq \kappa_f^2 \geq \dots \kappa_f^\ell \geq \dots,$$

and set

$$\kappa_f^\infty := \lim_{\ell \to \infty} \kappa_f^\ell.$$

Proposition

If $\kappa_f^\infty > 0$, then $f(M) \subset \mathcal{E}_{M'}$, i.e. f is a collapsing map.

Proposition

If $\kappa_f^\infty > 0$, then $f(M) \subset \mathcal{E}_{M'}$, i.e. f is a collapsing map.

- To get finite jet determination, we next need a FIXED integer e_0 such that any non-collapsing map germ f satisfies $\kappa_f^{e_0} = 0$.

So when does $(\kappa_f^\ell)_\ell$ stagnate?

Proposition

If $\kappa_f^\infty > 0$, then $f(M) \subset \mathcal{E}_{M'}$, i.e. f is a collapsing map.

- To get finite jet determination, we next need a FIXED integer e_0 such that any non-collapsing map germ f satisfies $\kappa_f^{e_0} = 0$.

So when does $(\kappa_f^\ell)_\ell$ stagnate?

Proposition (MP)

Let $p \in M$ and $f \in \mathcal{F}_p(M, M')$. Assume that there exists a positive integer s such that the sequence $(\kappa_f^\ell)_{\ell \in \mathbb{Z}_+}$ stagnates at some level $\kappa > 0$, that is, $\kappa_f^{\ell+s} = \kappa_f^\ell = \kappa$ for some ℓ. Then there exists a neighbourhood M_p of p in M, and an open dense subset of $\widetilde{M}_p$ of M_p, and a CR family $(\Gamma_\xi)_{\xi \in \widetilde{M}_p}$ of κ-dimensional complex submanifolds of $\mathbb{C}^{N'}$, such that each Γ_ξ is tangent to M' up to order s at $f(\xi)$.

A sufficient condition avoiding the previous situation is given in the following:

Corollary

Assume that the order of contact of (non-singular) complex curves with $M' = \{\rho = 0\}$ is uniformly bounded. Then there is a fixed integer e_0 such that any map germ f satisfies $\kappa_f^{e_0} = 0$.

A sufficient condition avoiding the previous situation is given in the following:

Corollary

Assume that the order of contact of (non-singular) complex curves with $M' = \{\rho = 0\}$ is uniformly bounded. Then there is a fixed integer e_0 such that any map germ f satisfies $\kappa_f^{e_0} = 0$.

For non-collapsing map germs, one may derive a better sufficient condition. The following notion will be very useful.

A sufficient condition avoiding the previous situation is given in the following:

Corollary

Assume that the order of contact of (non-singular) complex curves with $M' = \{\rho = 0\}$ is uniformly bounded. Then there is a fixed integer e_0 such that any map germ f satisfies $\kappa_f^{e_0} = 0$.

For non-collapsing map germs, one may derive a better sufficient condition. The following notion will be very useful.

Definition

Let ρ be a polynomial map as above. We say that ρ is of quasi-finite type if there exists an integer k_0 such that if $\gamma\colon (\mathbb{C}, 0) \to \mathbb{C}^{n'}$ is a non-singular holomorphic map with $\gamma(0) \in M'$ and $\gamma'(0) \neq 0$ satisfying $\nu_0(\rho \circ \gamma) > k_0$ then there exists a non-singular holomorphic curve $\Gamma \subset M'$ passing through $\gamma(0)$.

Lemma

ρ is of quasi-finite type if and only if

$$\sup_{q\in M'\setminus\mathcal{E}^1_{M'}} \sup\{\ \nu_0(\rho\circ\gamma);\gamma(0)=q\}<\infty$$

where $\mathcal{E}^1_{M'}$ is the set of points in M' through which there passes a non-singular complex curve contained in M'.

Lemma

ρ is of quasi-finite type if and only if

$$\sup_{q\in M'\setminus\mathcal{E}^1_{M'}} \sup\{\ \nu_0(\rho\circ\gamma); \gamma(0) = q\} < \infty$$

where $\mathcal{E}^1_{M'}$ is the set of points in M' through which there passes a non-singular complex curve contained in M'.

A consequence of the Proposition (MP) is:

Corollary

If ρ is of quasi-finite type, then there is a fixed integer e_0 such that any non-collapsing map germ f satisfies $\kappa_f^{e_0} = 0$.

Theorem (L-Mir-Rond (2023))

Any polynomial map is of quasi-finite type. (More generally, this is also true for Nash maps.)

Theorem (L-Mir-Rond (2023))

Any polynomial map is of quasi-finite type. (More generally, this is also true for Nash maps.)

This theorem, together with previous considerations, implies that there is a fixed finite family of universal systems satisfied by all non-collapsing map germs and for which the degeneracy is always zero. Theorem (A) concludes the whole proof.

Proposition (MP) can be used to establish further unique jet determination results for sheaves of collapsing maps as well, which occurs e.g. when $\mathcal{E}_{M'} = M'$.

Proposition (MP)

Let $p \in M$ and $f \in \mathcal{F}_p(M, M')$. Assume that there exists a positive integer s such that the sequence $(\kappa_f^\ell)_{\ell \in \mathbb{Z}_+}$ stagnates at some level $\kappa > 0$, that is, $\kappa_f^{\ell+s} = \kappa$ for some ℓ. Then there exists a neighbourhood M_p of p in M, and an open dense subset of $\widetilde{M}_p$ of M_p, and a CR family $(\Gamma_\xi)_{\xi \in \widetilde{M}_p}$ of κ-dimensional complex submanifolds of $\mathbb{C}^{N'}$, such that each Γ_ξ is tangent to M' up to order s at $f(\xi)$.

If the previous conclusion happens from some map f belonging to some sheaf $\mathcal{T}$, we shall say that there is a germ of an s-approximate CR $\mathcal{T}$-deformation from M into M'.

The following more detailed theorem follows from the proof.

Theorem

Let M and M' be as above and $\mathcal{T}$ a subsheaf of $\mathcal{F}(M, M')$ and suppose that there is no germ of a s_0-approximate CR $\mathcal{T}$-deformation from M into M' for some $s_0 \in \mathbb{Z}_+$. Then $\mathcal{T}$ satisfies the strong finite jet determination property.

The following more detailed theorem follows from the proof.

Theorem

Let M and M' be as above and $\mathcal{T}$ a subsheaf of $\mathcal{F}(M, M')$ and suppose that there is no germ of a s_0-approximate CR $\mathcal{T}$-deformation from M into M' for some $s_0 \in \mathbb{Z}_+$. Then $\mathcal{T}$ satisfies the strong finite jet determination property.

This result was previously proved by L-Mir in 2022 for $s_0 = 2$ and has direct applications to e.g. mappings into boundaries of classical domains. We conclude with such an application which follows from the non existence of 2-approximate CR deformations of transversal CR maps.

Boundaries of classical domains

$$D_I^{m,n} = \{Z \in \mathbb{C}^{m\times n} \colon \mathbb{I}_m - ZZ^* > 0\}$$
$$D_{II}^m = \left\{Z \in \mathbb{C}^{m\times m} \colon Z^T = -Z, \mathbb{I}_m - Z^*Z > 0\right\}$$
$$D_{III}^m = \left\{Z \in \mathbb{C}^{m\times m} \colon Z^T = Z, \mathbb{I}_m - Z^*Z > 0\right\}$$
$$D_{IV}^m = \left\{z \in \mathbb{C}^{m} \colon z^*z < 1, 1 + |z^Tz|^2 - 2z^*z > 0\right\}.$$

The regular parts of the boundary of such domains are uniformly pseudoconvex i.e. pseudoconvex with a Levi-form of constant rank. In particular, they are foliated by complex submanifolds (the integral manifolds of the distribution given by the Levi kernels).

Corollary (L-Mir (2022))

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, and $M \subset \partial\Omega$ be a connected real-analytic minimal hypersurface of $\mathbb{C}^n$, with generically n_+ positive Levi eigenvalues. Let Ω' be one of the four types of classical domains $D_I^{m,n}, D_{II}^m, D_{III}^m, D_{IV}^m$ satisfying:

- *if $\Omega' = D_I^{m,n}$, $m, n \geq 2$, and, $m + n - 4 < n_+ \leq m + n - 2$;*
- *if $\Omega' = D_{II}^m$, $m \geq 4$, and, $2m - 8 < n_+ \leq 2m - 4$;*
- *if $\Omega' = D_{III}^m$, $m \geq 2$, and $n_+ = m - 1$;*
- *if $\Omega' = D_{IV}^m$, $m \geq N$.*

There exists a locally bounded map $\ell \colon M \to \mathbb{Z}_+$, such that given two proper holomorphic mappings $H, G \colon \Omega \to \Omega'$, extending smoothly up to some point $p \in M$ with $H(p)$ in the regular part of $\partial\Omega'$, if $j_p^{\ell(p)} H = j_p^{\ell(p)} G$, then it follows that $H = G$.

Corollary (L-Mir (2022))

Let $\Omega \subset \mathbb{C}^n$ be a pseudoconvex domain, and $M \subset \partial\Omega$ be a connected real-analytic minimal hypersurface of $\mathbb{C}^n$, with generically n_+ positive Levi eigenvalues. Let Ω' be one of the four types of classical domains $D_I^{m,n}, D_{II}^m, D_{III}^m, D_{IV}^m$ satisfying:

- *if $\Omega' = D_I^{m,n}$, $m, n \geq 2$, and, $m + n - 4 < n_+ \leq m + n - 2$;*
- *if $\Omega' = D_{II}^m$, $m \geq 4$, and, $2m - 8 < n_+ \leq 2m - 4$;*
- *if $\Omega' = D_{III}^m$, $m \geq 2$, and $n_+ = m - 1$;*
- *if $\Omega' = D_{IV}^m$, $m \geq N$.*

There exists a locally bounded map $\ell \colon M \to \mathbb{Z}_+$, such that given two proper holomorphic mappings $H, G : \Omega \to \Omega'$, extending smoothly up to some point $p \in M$ with $H(p)$ in the regular part of $\partial\Omega'$, if $j_p^{\ell(p)} H = j_p^{\ell(p)} G$, then it follows that $H = G$.

It is interesting to note that the range of dimensions given above is also sharp in order for the finite jet determination property to hold.

Thank you for your attention!