Second jet determination for CR mappings

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Complex Analysis, Geometry and Dynamics
Portorož, June 5–9, 2023

A CR mapping is a diffeomorphism between two real manifolds in complex space that satisfies tangential Cauchy-Riemann equations. We are concerned with the problem whether a CR mapping is uniquely determined by its finite jet at a point. This problem has been popular since 1970-s and the number of publications on the matter is enormous. Nevertheless, natural fundamental questions have remained open. I will present a solution to a version of the problem and discuss old and new results.

- Conditions on the Levi form
- Infinitesimal automorphisms of quadrics
- Finite jet determination
- 2-jet determination
- Examples

Equation of a generic manifold

Let $M \subset \mathbb{C}^n$ be a generic submanifold of real codimension $\operatorname{cod} M = k$ and CR dimension $\dim_{CR} M = m = n - k$. We introduce coordinates $(z, w) \in \mathbb{C}^n$, $z \in \mathbb{C}^m$, $w = u + iv \in \mathbb{C}^k$, so that M has a local equation

$$v = h(z, u),$$

where $h = (h_1, \ldots, h_k)$ is a smooth real vector function with $h(0) = 0$, $dh(0) = 0$.

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Then $T_0(M)$ and $T_0^c(M)$ have have equations respectively $v = 0$ and $w = 0$.

Equation of a generic manifold

We choose the coordinates so that the equation of M takes the form

$$v = h(z, u) = F(z, z) + O(|z|^3 + |u|^3).$$

Here

$$F = (F_1, \ldots, F_k), \quad F_j(z, z) = \langle A_j z, \overline{z} \rangle, \quad \langle a, b \rangle = \sum a_l b_l;$$

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A_j-s are Hermitian matrices.

The matrices A_j can be regarded as the components of the vector valued Levi form of M at 0.

Conditions on the Levi form

- We say M is (Levi) nondegenerate at 0 if
  (a) the matrices A_j are linearly independent and
  (b) $F(z,\zeta) = 0$ for all $z \in \mathbb{C}^m$ implies $\zeta = 0$.

  If this condition is not fulfilled, then the quadratic manifold $v = F(z,z)$ has infinite dimensional set of CR maps to itself.

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- We say M is strongly nondegenerate at 0 if M is nondegenerate and there is $c \in \mathbb{R}^k$ such that $\det\left(\sum c_j A_j\right) \neq 0$. This condition implies that M lies on a Levi nondegenerate hypersurface.

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- We say M is strongly nondegenerate at 0 if M is nondegenerate and there is $c \in \mathbb{R}^k$ such that $\det\left(\sum c_j A_j\right) \neq 0$. This condition implies that M lies on a Levi nondegenerate hypersurface.

- We say M is strongly pseudoconvex at 0 if there is $c \in \mathbb{R}^k$ such that $\sum c_j A_j > 0$. This condition implies that M lies on a strognly pseudoconvex hypersurface.

CR mappings

Let M_1 and M_2 be CR manifolds. A C^1 mapping $f : M_1 \to M_2$ is called a CR mapping or a CR map if $df|_{T^c(M_1)}$ is a $\mathbb{C}$-linear mapping $T^c(M_1) \to T^c(M_2)$.

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If a CR mapping is a diffeomorphism, then it is called a CR diffeomorphism. Clearly, if $f : M_1 \to M_2$ is a CR diffeomorphism of generic manifolds in $\mathbb{C}^n$, then M_1 and M_2 should have the same dimension and CR dimension. We will consider only CR diffeomorphisms and will call them just CR mappings.

Finite jet determination

We are concerned with the problem whether a CR mapping is uniquely determined by its finite jet at a point, which is referred to as finite jet determination. This problem has been a subject of work by many authors (Baouendi, Beloshapka, Bertrand, Ebenfelt, Ezhov, Han, Kim, Lamel, Merker, Meylan, Rothschild, Schmalz, Sukhov, Zaitsev, ...).

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Beloshapka (1988) proved that a real analytic CR automorphism of a real analytic nondegenerate CR manifold is determined by its finite jet at a point.

2-jet determination

Tanaka (1967) gave a solution to the CR equivalence problem for nondegenerate CR manifolds of codimensions $k = 1, m^2 - 1, m^2$. His result implies 2-jet determination for real analytic CR mappings of real analytic nondegenerate manifolds of said codimensions.

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Tanaka's result for a real hypersurface ($k = 1$) was later rediscovered by Chern and Moser (1974).

2-jet determination

Bertrand, Blanc-Centi and Meylan (2019-2020), prove 2-jet determination for C^3-smooth CR automorphisms of C^4-smooth generic nondegenerate manifold M with additional condition that the authors call D-nondegenerate. In particular, it implies that there is $z \in \mathbb{C}^m$ such that the vectors $\{A_j z : 1 \le j \le k\}$ are $\mathbb{R}$-linearly independent. This condition is quite restrictive, in particular, it implies that $\operatorname{cod} M \le 2 \dim_{CR} M$, whereas the dimension of the space of all Hermitian forms on $\mathbb{C}^m$ is equal to m^2.

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Both results above were obtained by using the invariantness of stationary discs.

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We present a sufficient condition for 2-jet determination that implies all affirmative results on 2-jet determination mentioned above, that is, for strictly pseudoconvex, D-nondegenerate and codimension ≤ 3 CR manifolds. Our approach is based on infinitesimal automorphisms of real quadrics.

Infinitesimal automorphisms of quadrics

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Let M be a nondegenerate quadric defined as before by the equations

$$v = F(z, z), \quad z \in \mathbb{C}^m, \quad w = u + iv \in \mathbb{C}^k.$$

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$$F = (F_1, \ldots, F_k), \quad F_j(z, z) = \langle A_j z, \bar{z} \rangle, \quad \langle a, b \rangle = \sum a_l b_l.$$

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$$F = (F_1, \ldots, F_k), \quad F_j(z, z) = \langle A_j z, \bar{z} \rangle, \quad \langle a, b \rangle = \sum a_l b_l.$$

Let G be the group of all CR-mappings (CR-automorphisms) $M \to M$. Then G is a finite dimensional Lie group and its Lie algebra $\mathfrak{g}$ is the set of all infinitesimal automorphisms of M. The dimension of G has an estimate depending on m and k. (Beloshapka 1988, Tumanov 1988, Isaev and Kaup 2012, ...)

It turns out that all elements of G and $\mathfrak{g}$ are respectively rational and polynomial. In particular, every vector field $X \in \mathfrak{g}$ has the form

$$X = 2\mathrm{Re}\left(\sum f_j \frac{\partial}{\partial z_j} + \sum g_\ell \frac{\partial}{\partial w_\ell}\right) = 2\mathrm{Re}\left(f \frac{\partial}{\partial z} + g \frac{\partial}{\partial w}\right) =: (f, g),$$

where f and g are polynomial vector functions in z and w that satisfy the equation

$$\mathrm{Im}\,(g - 2iF(f, z)) = 0, \quad (z, w) \in M.$$

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$$\mathrm{Im}\,(g - 2iF(f, z)) = 0, \quad (z, w) \in M.$$

This equation implies

$$\deg_z f \leq 2, \quad \deg_z g \leq 1.$$

The graded algebra $\mathfrak{g}$

We give the variables and differentiations $z_j, w_j, \partial/\partial z_j, \partial/\partial w_j$ the weights 1,2,-1,-2 respectively. Let $\mathfrak{g}_p$ be the set of vector fields $X \in \mathfrak{g}$ with weighted homogeneous degree $p \in \mathbb{Z}$. Then

$$\mathfrak{g} = \sum_{p=-2}^{\infty} \mathfrak{g}_p$$

is a graded Lie algebra, that is, $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$. The terms $\mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1}$ have the same form for all quadrics:

$$\mathfrak{g}_{-2} = \{b\frac{\partial}{\partial w} : b \in \mathbb{R}^k\}$$

$$\mathfrak{g}_{-1} = \{a\frac{\partial}{\partial z} + 2iF(z,a)\frac{\partial}{\partial w} : a \in \mathbb{C}^m\}.$$

For $p \geq 0$, the structure of $\mathfrak{g}_p$ depends significantly on F.

Since F is nondegenerate, it follows that each vector $\xi \in \mathfrak{g}_p$ is uniquely determined by the map $\mathrm{ad}\,\xi : \mathfrak{g}_{-1} \to \mathfrak{g}_{p-1}$, here $(\mathrm{ad}\,\xi)(\eta) = [\xi, \eta]$.

In particular, if $\mathfrak{g}_p = 0$, then $\mathfrak{g}_q = 0$ for all $q > p$.

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In particular, if $\mathfrak{g}_p = 0$, then $\mathfrak{g}_q = 0$ for all $q > p$.

Thus, the algebra $\mathfrak{g}$ is the Tanaka prolongation of $\mathfrak{g}_{-2} + \mathfrak{g}_{-1}$, that is, the maximal graded Lie algebra with the above unique determination property.

Finite jet determination

Let M be a nondegenerate CR manifold with equation

$$v = h(z, u) = F(z, z) + O(|z|^3 + |u|^3),$$

and let M_0 be the corresponding quadric with equation

$$v = F(z, z).$$

Let $\mathfrak{g}$ be the graded Lie algebra of infinitesimal automorphisms of M_0. Finite dimensionality of $\mathfrak{g}$ implies finite jet determination for CR mappings of M.

Theorem

Let M, M' be smooth non-degenerate CR manifolds defined as above. Suppose $\mathfrak{g}_p = 0$ for some $p > 0$. Then every germ at 0 of a smooth CR diffeomorphism $\Phi = (f, g) : M \to M'$ with $\Phi(0) = 0$ is uniquely determined by the jets of f and g at 0 of weights respectively p and $p + 1$.

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Corollary

Let M, M' be smooth non-degenerate CR manifolds defined as above. Suppose $\mathfrak{g}_3 = 0$. Then every germ at 0 of a smooth CR diffeomorphism $\Phi : M \to M'$ is uniquely determined by the 2-jet of Φ at 0. Conversely, if $\mathfrak{g}_3 \neq 0$, then there exists a CR diffeomorphism $\Phi : M_0 \to M_0$, $\Phi \neq \mathrm{id}$, whose 2-jet at 0 is the identity.

Beloshapka (1988) obtained the real analytic versions.

Proof

Following Moser (1974) and Beloshapka (1988), we expand the equations of M and M' and the CR mapping $\Phi = (f, g)$ into Taylor series with remainders and represent them as sums of weighed homogeneous components.

$$\begin{aligned} v &= h(z, u) = F + h_3 + \dots \\ v' &= h'(z', u') = F' + h'_3 + \dots \\ z' &= f(z, w) = f_1 + f_2 + \dots \\ w' &= g(z, w) = g_2 + g_3 + \dots \end{aligned}$$

Since the derivative of Φ maps the complex tangent plane $w = 0$ to the plane $w' = 0$, we have $g_1 = 0$. By linear transformations of z and w, we can put $f_1 = z$, $g_2 = w + P(z)$, where P is a quadratic polynomial, but one can see that $P = 0$. Also, one can see that $F' = F$.

By plugging z' and w' in terms of z and $w = u + ih(z, u)$ in the equation of M' we obtain an equation for the component (f_{p+1}, g_{p+2}) of Φ:

$$\mathrm{Im}\,(g_{p+2} - 2iF(f_{p+1}, z))|_{w=u+iF(z,z)} = \dots,$$

here the dots mean terms that include only f_{q+1} and g_{q+2} with $q < p$.

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Note that the corresponding homogeneous equation describes $(f_{p+1}, g_{p+2}) \in \mathfrak{g}_p$. Since $\mathfrak{g}_p = 0$, the component (f_{p+1}, g_{p+2}) is uniquely determined by the components of Φ of lower weighted degree.

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Since $\mathfrak{g}_q = 0$ for all $q > p$, we can successively uniquely determine all components (f_{q+1}, g_{q+2}) for $q > p$. This completes the proof in the real analytic case.

In the smooth case, we can apply the above argument to pairs of points $(z, w) \in M$, $(z', w') = \Phi(z, w) \in M'$. This results in an overdetermined PDE on Φ whose solution is uniquely determined by a jet at just one point. This completes the proof.

Vanishing of $\mathfrak{g}_3$

Let M, F be as above. Let $S \subset \mathbb{C}^m$ be a set. We define

$$S^F = \{z \in \mathbb{C}^m : \forall z' \in S, F(z, z') = 0\}.$$

We also define a complex subspace

$$T(z) = \{p \in \mathbb{C}^m : \exists q \in \mathbb{C}^m : F(z, p) + F(q, z) = 0\}.$$

We say that M and F are T-nondegenerate if for generic z (that is, for all z in an open dense set) we have

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Theorem

Let M, M' be smooth T-nondegenerate CR manifolds. Then $\mathfrak{g}_3 = 0$. Hence, 2-jet determination for smooth CR diffeomorphisms $M \to M'$ takes place.

Other conditions

Recall that F is strongly pseudoconvex if there is $c \in \mathbb{R}^k$ such that $A = \sum_{j=1}^{k} c_j A_j > 0$, positive definite. We proved (2020) 2-jet determination for strongly pseudoconvex CR manifolds. We recover this result here.

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- Let the form F be strongly pseudoconvex. Then F is T-nondegenerate, hence $\mathfrak{g}_3 = 0$.

For a strongly pseudoconvex F, we have $F(z, z) = 0$ only if $z = 0$. Hence the claim follows from a more general one.

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For a strongly pseudoconvex F, we have $F(z, z) = 0$ only if $z = 0$. Hence the claim follows from a more general one.

- Suppose for generic $a \in \mathbb{C}^m$, we have $(a^F)^F \cap a^F = 0$. Then F is T-nondegenerate.

Indeed, $T(a)^F \subset (a^F)^F \cap a^F = 0$, and the claim follows.

Other conditions

Bertrand, Blanc-Centi, and Meylan (2019) prove 2-jet determination for so called fully nondegenerate CR manifolds in the smooth case. Their condition, in particular, implies that there is $a \in \mathbb{C}^m$ such that the vectors $(A_j a)_{j=1}^k$ are $\mathbb{C}$-linear independent. We recover this result here.

- Suppose there is $a \in \mathbb{C}^m$ such that the vectors $(A_j a)_{j=1}^k$ are $\mathbb{C}$-linear independent. Then F is T-nondegenerate.

Indeed, in this case for generic $a \in \mathbb{C}^m$, we have $T(a) = \mathbb{C}^m$, and $T(a)^F = 0$.

Other conditions

Beloshapka (2022) proves a sufficient condition for $\mathfrak{g}_3 = 0$ that, in particular, includes the hypothesis that there is $a \in \mathbb{C}^m$ such that the vectors $(A_j a)_{j=1}^k$ span $\mathbb{C}^m$ over $\mathbb{C}$. We observe that this hypothesis alone suffices for $\mathfrak{g}_3 = 0$.

- Suppose there is $a \in \mathbb{C}^m$ such that the vectors $(A_j a)_{j=1}^k$ span $\mathbb{C}^m$ over $\mathbb{C}$. Then F is T-nondegenerate.

Indeed, for generic $a \in \mathbb{C}^m$, we have $a^F = 0$, hence $T(a)^F = 0$.

Other conditions: D-nondegeneracy

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Following Bertrand and Meylan (2021), we introduce matrices

$$D = (A_1 a, \ldots, A_k a), \quad B = D^* A^{-1} D.$$

The form F is called D-nondegenerate if there exist $c \in \mathbb{R}^k$ and $a \in \mathbb{C}^m$ such that the matrix $\mathrm{Re}\, B := \frac{1}{2}(B + \overline{B})$ is nonsingular. Bertrand and Meylan (2021) prove 2-jet determination for D-nondegenerate CR manifolds in the smooth case.

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We show

- If F is D-nondegenerate, then F is T-nondegenerate.

Indeed, one can see that $\det \mathrm{Re}\, B \neq 0$ implies $(a^{\mathrm{Re}\, F})^{\mathrm{Re}\, F} \cap a^{\mathrm{Re}\, F} = 0$, and $T(a)^F \subset (a^{\mathrm{Re}\, F})^{\mathrm{Re}\, F} \cap a^{\mathrm{Re}\, F}$.

Codimension $k \leq 3$

By Tanaka (1967) and Chern and Moser (1974), 2-jet determination holds for codimension $k = 1$. Blanc-Centi and Meylan (2022) prove 2-jet determination for holomorphic CR mappings for $k = 2$. Beloshapka (2022) proves that $\mathfrak{g}_3 = 0$ for $k = 3$. We recover these results here by proving the following.

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- Let F be a nondegenerate quadric with $k \leq 3$. Then F is T-nondegenerate, hence $\mathfrak{g}_3 = 0$.

Proof of Main result

An element $(f, g) \in \mathfrak{g}_3$ has the following form

$$f(z, w) = A(z, z, w) + B(w, w),$$
$$g(z, w) = 2iF(z, B(\overline{w}, \overline{w})).$$

Here A and B are complex multilinear forms such that A is symmetric in the first two arguments and B is symmetric. They are characterized by the following equations:

$$F(A(z, z, F(z, b)), b) = 0, \tag{1}$$
$$F(A(z, z, w), b) = 4iF(z, B(\overline{w}, F(b, z))). \tag{2}$$

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We would like to show that $A = 0$ and $B = 0$. The difficulty is that the equations have repeated arguments.

The main tool is the following.

Lemma

Let $\phi : \mathbb{C}^k \to \mathbb{C}^m$ be a C^1 mapping. Suppose for every $a, b \in \mathbb{C}^m$, we have $F(\phi(F(a,b)), b) = 0$. Then for every $a \in \mathbb{C}^m$, we have $\phi(F(a,a)) \in T(a)^F$. Hence, if F is T-nondegenerate, then for every $a \in \mathbb{C}^m$, we have $\phi(F(a,a)) = 0$.

We first plug $w = F(z, b)$ in (2) and using (1) we obtain

$$F(z, B(F(b, z), F(b, z))) = 0. \tag{3}$$

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Put $\phi(w) = B(w, w)$. Then by (3) we have $F(\phi(F(b, z)), z) = 0$ for all $b, z \in \mathbb{C}^m$.

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Since F is T-nondegenerate, by Lemma, we have $\phi(F(b, b)) = 0$, that is,

$$B(F(b, b), F(b, b)) = 0.$$

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By polarization

$$B(F(b, z), F(b, z)) = 0. \tag{4}$$

Plugging $w = F(a, b)$ in (2), using (4) we obtain

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Applying Lemma with $\phi(w) = A(z, z, w)$, we obtain

$$A(z, z, F(a, a)) = 0,$$

hence $A = 0$.

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Similarly, applying Lemma one more time, we finally show that $B(c, F(b, z)) = 0$, hence $B = 0$.

This completes the proof.

Example 1 (Meylan)

Francine Meylan found an example of a (strictly) nondegenerate quadric for which $\mathfrak{g}_4 \neq 0$. Here $m = 4, k = 5$, $F = (F_1, \ldots, F_5)$.

$$\begin{aligned} F_1(z, z) &= \mathrm{Re}\,(z_1\overline{z}_3 + z_2\overline{z}_4), \\ F_2(z, z) &= |z_1|^2, \\ F_3(z, z) &= |z_2|^2, \\ F_4(z, z) &= \mathrm{Re}\,(z_1\overline{z}_2), \\ F_5(z, z) &= \mathrm{Im}\,(z_1\overline{z}_2). \end{aligned}$$

In this example, $T(z)^F \neq 0$ for all $z \neq 0$, that is F is not T-nondegenerate, all stationary discs are defective, and 2-jet determination fails.

Example 2

Let $m = 4, k = 4$, and $F = (F_1, F_2, F_3, F_4)$ from Example 1. Then F is *not* T-nondegenerate, but one can see that $\mathfrak{g}_3 = 0$, so 2-jet determination takes place.

Example 2

Let $m = 4, k = 4$, and $F = (F_1, F_2, F_3, F_4)$ from Example 1. Then F is *not* T-nondegenerate, but one can see that $\mathfrak{g}_3 = 0$, so 2-jet determination takes place.

This example shows that T-nondegeneracy is not necessary for $\mathfrak{g}_3 = 0$ to hold.

Example 3

Let $m = 4, k = 3$, and $F = (F_1, F_2, F_3)$ from Example 1. Since $k = 3$, the form F is T-nondegenerate, $\mathfrak{g}_3 = 0$, so 2-jet determination takes place.

Example 3

Let $m = 4$, $k = 3$, and $F = (F_1, F_2, F_3)$ from Example 1. Since $k = 3$, the form F is T-nondegenerate, $\mathfrak{g}_3 = 0$, so 2-jet determination takes place.

However, one can see that F is *not* D-nondegenerate and $(a^F)^F \cap a^F \neq 0$ for generic a.

Example 4

Monomial quadric. We call F monomial if all components of F have the form $\mathrm{Re}\,(z_p\overline{z_q})$ (in particular, $|z_p|^2$) or $\mathrm{Im}\,(z_p\overline{z_q})$ ($p \neq q$), and there are no repeated components. A monomial quadric F is nondegenerate iff each variable z_p occurs in at least one component of F. For such a quadric, the codimension k can be any integer $\frac{m}{2} \leq k \leq m^2$.

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One can see that if $a \in \mathbb{C}^m$ has no zero components, then $(a^F)^F \cap a^F = 0$. Hence F is T-nondegenerate. However, F can not be D-nondegenerate if $k > 2m$.

Example 5

Monomial antisymmetric quadric of an odd dimension. This is a special case of the previous example in which $m \geq 3$ is an odd integer, and all components of F have the from $\mathrm{Im}\,(z_p \overline{z_q})$. Then $k \leq \frac{m(m-1)}{2}$.

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Note that F is not strongly nondegenerate, hence not D-nondegenerate. Indeed, the matrices of the components of F are antisymmetric, and every linear combination of them is an antisymmetric matrix of an odd order, hence singular.

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Note that F is not strongly nondegenerate, hence not D-nondegenerate. Indeed, the matrices of the components of F are antisymmetric, and every linear combination of them is an antisymmetric matrix of an odd order, hence singular.

As a special case of the previous example, if F is nondegenerate, then F is T-nondegenerate, and 2-jet determination holds. However, this result would be difficult to obtain by means of stationary discs because F is not strongly nondegenerate.

Thank you!