Metric properties and geometry of domains in $\mathbb{C}^d$

Hervé Gaussier, Université Grenoble Alpes
Joint work with Andrew Zimmer

Complex Analysis, Geometry and Dynamics

Portoroz 2023

Some characterizations ... before definitions

Theorem. [Hawley 1953, Igusa 1954]. A complete simply connected Kähler mnfld. (M, g) with $H(g) \equiv cst$. is biholomorphic to $\mathbb{CP}^d$, $\mathbb{B}^d$ or $\mathbb{C}^d$.

Positive curvature

Theorem [Mori 1979, Siu-Yau 1980] Every compact Kähler manifold of **positive hol. bisectional curvature** is biholomorphic to the complex projective space.

Conjecture (Yau). A complete noncompact Kähler manifold with positive hol. bisectional curvature is biholomorphic to $\mathbb{C}^d$*.*

Some characterizations ... before definitions

Theorem. [Hawley 1953, Igusa 1954]. A complete simply connected Kähler mnfld. (M, g) with $H(g) \equiv cst.$ is biholomorphic to $\mathbb{CP}^d$, $\mathbb{B}^d$ or $\mathbb{C}^d$.

Positive curvature

Theorem [Mori 1979, Siu-Yau 1980] Every compact Kähler manifold of **positive hol. bisectional curvature** is biholomorphic to the complex projective space.

Conjecture (Yau). A complete noncompact Kähler manifold with positive hol. bisectional curvature is biholomorphic to $\mathbb{C}^d$.

Flat curvature

Theorem [Siu-Yau, 1977] If (M, g) is a simply connected, complete, Kähler manifold of complex dimension n with $-\frac{A}{r^{2+\varepsilon}} \leq sec(g) \leq 0$, where A, $\varepsilon > 0$ and r is the distance from a fixed point of M, then M is biholomorphic to $\mathbb{C}^d$.

Some characterizations ... before definitions

Theorem. [Hawley 1953, Igusa 1954]. A complete simply connected Kähler mnfld. (M, g) with $H(g) \equiv cst$. is biholomorphic to $\mathbb{CP}^d$, $\mathbb{B}^d$ or $\mathbb{C}^d$.

Positive curvature

Theorem [Mori 1979, Siu-Yau 1980] Every compact Kähler manifold of **positive hol. bisectional curvature** is biholomorphic to the complex projective space.

Conjecture (Yau). A complete noncompact Kähler manifold with positive hol. bisectional curvature is biholomorphic to $\mathbb{C}^d$.

Flat curvature

Theorem [Siu-Yau, 1977] If (M, g) is a simply connected, complete, Kähler manifold of complex dimension n with $-\frac{A}{r^{2+\varepsilon}} \leq sec(g) \leq 0$, where A, $\varepsilon > 0$ and r is the distance from a fixed point of M, then M is biholomorphic to $\mathbb{C}^d$.

Negative curvature

Generically, 2 small $\mathcal{C}^\infty$ deformations of $\mathbb{B}^d \subset \mathbb{C}^d$ are not bihol.
They have neg. pinched hol. bisect. curv. converging asymptotically to that of $\mathbb{B}^d$.

Question. What information do strong constraints on complete metrics with negative curvature give on the boundary (at infinity) of a complex manifold ?

• **Let g complete Kähler metric on M_d, $H(g)(z) \rightarrow_{z\rightarrow\infty} -c < 0$ sufficiently fast. Then $\tilde{M} \simeq \mathbb{B}^d \subset \mathbb{C}^d$.**

Question. What information do strong constraints on complete metrics with negative curvature give on the boundary (at infinity) of a complex manifold ?

• **Let g complete Kähler metric on M_d, $H(g)(z) \to_{z\to\infty} -c < 0$ sufficiently fast. Then $\tilde{M} \simeq \mathbb{B}^d \subset \mathbb{C}^d$.**

Theorem[G.-Zimmer, 2022]

Let X_d be a Stein manifold with $d \geq 2$ and g_0 is a complete Kähler metric on X. If there exists a compact set $K \subset X$ such that g_0 has constant negative holomorphic sectional curvature on $X \setminus K$, then the universal cover of X is biholomorphic to the unit ball in $\mathbb{C}^d$.

Question. What information do strong constraints on complete metrics with negative curvature give on the boundary (at infinity) of a complex manifold ?

- **Let g complete Kähler metric on M_d, $H(g)(z) \to_{z\to\infty} -c < 0$ sufficiently fast. Then $\tilde{M} \simeq \mathbb{B}^d \subset \mathbb{C}^d$.**

Theorem[G.-Zimmer, 2022]

Let X_d be a Stein manifold with $d \geq 2$ and g_0 is a complete Kähler metric on X. If there exists a compact set $K \subset X$ such that g_0 has constant negative holomorphic sectional curvature on $X \setminus K$, then the universal cover of X is biholomorphic to the unit ball in $\mathbb{C}^d$.

- **M_d complete (Kobayashi) hyperbolic complex manifold. Assume k_M is Kähler. Then $\tilde{M} \simeq \mathbb{B}^d$.**

Question. What information do strong constraints on complete metrics with negative curvature give on the boundary (at infinity) of a complex manifold ?

- **Let g complete Kähler metric on M_d, $H(g)(z) \to_{z\to\infty} -c < 0$ sufficiently fast. Then $\tilde{M} \simeq \mathbb{B}^d \subset \mathbb{C}^d$.**

Theorem[G.-Zimmer, 2022]

Let X_d be a Stein manifold with $d \geq 2$ and g_0 is a complete Kähler metric on X. If there exists a compact set $K \subset X$ such that g_0 has constant negative holomorphic sectional curvature on $X \setminus K$, then the universal cover of X is biholomorphic to the unit ball in $\mathbb{C}^d$.

- **M_d complete (Kobayashi) hyperbolic complex manifold. Assume k_M is Kähler. Then $\tilde{M} \simeq \mathbb{B}^d$.**

Theorem[G.-Zimmer, 2022]

True if M_d is a bounded strictly pseudoconvex domain in $\mathbb{C}^d$.

For M cplex mnfld., $z \in M$, $v \in T_z M$, we define :

- **The Kobayashi pseudometric**

$$k_M(z;v) = \inf\left\{\alpha > 0 / \ \exists f : \mathbb{D} \xrightarrow{\text{hol.}} M,\ f(0) = z,\ f'(0)\cdot\alpha = v\right\}.$$

- **The Carathéodory pseudometric**

$$c_M(z;v) = \sup\left\{|f'(z)(v)|,\ f : M \xrightarrow{\text{hol.}} \mathbb{D},\ f(z) = 0\right\}.$$

For M cplex mnfld., $z \in M$, $v \in T_z M$, we define :

- **The Kobayashi pseudometric**

$$k_M(z;v) = \inf\left\{\alpha > 0/\ \exists f : \mathbb{D} \xrightarrow{\text{hol.}} M,\ f(0) = z,\ f'(0)\cdot\alpha = v\right\}.$$

- **The Carathéodory pseudometric**

$$c_M(z;v) = \sup\left\{|f'(z)(v)|,\ f : M \xrightarrow{\text{hol.}} \mathbb{D},\ f(z) = 0\right\}.$$

Examples.

- All bounded domains in $\mathbb{C}^d$ are Carathéodory hyperbolic.
- $\mathbb{CP}^d \setminus (2d+1)$ hyperplanes in general position is Kobayashi hyperbolic but is not Carathéodory hyperbolic.

For M cplex mnfld., $z \in M$, $v \in T_z M$, we define :

- **The Kobayashi pseudometric**

$$k_M(z;v) = \inf\left\{\alpha > 0/\ \exists f : \mathbb{D} \xrightarrow{\text{hol.}} M,\ f(0) = z,\ f'(0)\cdot\alpha = v\right\}.$$

- **The Carathéodory pseudometric**

$$c_M(z;v) = \sup\left\{|f'(z)(v)|,\ f : M \xrightarrow{\text{hol.}} \mathbb{D},\ f(z) = 0\right\}.$$

Theorem [Stanton, 1983] Let M be a connected complete hyperbolic complex manifold. Assume that there is a point of M at which the Carathéodory and Kobayashi metrics are equal and that one of these metrics is Hermitian and of class $\mathcal{C}^\infty$. Then M is biholomorphically equivalent to the open unit ball.

Kähler metrics : D **pseudoconvex, bounded in** $\mathbb{C}^d$

- The Bergman metric (Bergman, Hörmander, Fefferman)

$$b_{i\bar{j}} = \frac{\partial^2 \ln K}{\partial z_i \partial \overline{z_j}}$$

- The Kähler-Einstein metric (Cheng-Yau, Mok-Yau)

  Solution of

$$\begin{cases} Det[g_{i\bar{j}}] = e^{(n+1)g}, & \text{on } D \\ g = +\infty & \text{on } \partial D. \end{cases}$$

Conj. (Cheng) *(D pcvex., $B_D = KE_D$) $\Rightarrow$ D homogeneous*

Curvature classification in complex dimension greater than one ?

Let (M, g) be a Kähler manifold, let $x \in M$.

Sectional curvature. For $X, Y \in T_x M \setminus \{0\}$:

$$sec(g)(v, w) =: R(v, w, v, w) / \left(g(X, X)g(Y, Y) - g(X, Y)^2\right),$$

where

$$\begin{cases} R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ R(X, Y, Z, W) = g(R(X, Y)Z, W) \end{cases}.$$

Curvature classification in complex dimension greater than one ?

Let (M, g) be a Kähler manifold, let $x \in M$.

Sectional curvature. For $X, Y \in T_x M \setminus \{0\}$:

$$sec(g)(v, w) =: R(v, w, v, w) / \left(g(X, X) g(Y, Y) - g(X, Y)^2 \right),$$

where

$$\begin{cases} R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ R(X, Y, Z, W) = g(R(X, Y)Z, W) \end{cases}.$$

Holomorphic (bi)sectional curvature. For $X, Y \in T_x M \setminus \{0\}$:

$$\mathbf{Bisec(g)(X, Y)} := R(g)(X, JX, Y, JY) / (g(X, X) g(Y, Y))$$

$$\mathbf{H(g)(X)} = Bisec(g)(X, X).$$

Negative hol. (bi)sectional curvature : some examples.

- **In the unit ball $\mathbb{B}^d \subset \mathbb{C}^d$**

  - "All" the metrics coincidence

  - $H(b_{\mathbb{B}^d}) = -2/(d+1)$ and $-2/(d+1) \leq Bisec(b_{\mathbb{B}^d}) \leq -1/(d+1)$

- **In the bidisk $\mathbb{D} \times \mathbb{D} \subset \mathbb{C}^2$**

  - $k_{\mathbb{D}\times\mathbb{D}}((z,w),(X,Y)) = \max(k_{\mathbb{D}}(z,X), k_{\mathbb{D}}(w,Y))$

  - $b_{\mathbb{D}\times\mathbb{D}}((z,w),(X,Y)) = \left((k_{\mathbb{D}}(z,X))^2 + (k_{\mathbb{D}}(w,Y))^2\right)^{1/2}$

  - $-1 \leq H(b_{\mathbb{D}\times\mathbb{D}}) \leq -1/2$ but $Bisec(b_{\mathbb{D}\times\mathbb{D}})(0)(e_1,e_2) = \mathbf{0}$

- **$D \subset\subset \mathbb{C}^d$, D str. pcv., $\partial D \in \mathcal{C}^\infty$**

  $(b_D = ke_D) \Rightarrow D \simeq \mathbb{B}^d$ [Huang-Xiao].

Negative hol. (bi)sectional curvature

Question 1. Which domains in $\mathbb{C}^d$ admit a complete Kähler metric with negative (bi)hol. curvature near the boundary ?

- Such a metric is bi-Lipschitz to the Kobayashi metric [Wu-Yau]

- $\exists\, D \subset\subset \mathbb{C}^3,\ \partial D \in \mathcal{C}^\infty, \partial D$ *of* $< \infty$ *type,* k_D *not bi-Lipschitz to a Riemannian metric (Herbort/Fornaess-Rong)*

[Wu-Yau] $\Rightarrow$ D does not admit any complete Kähler metric g, with

$$-A \leq H(g) \leq -B < 0 \text{ on } D.$$

Need some **CR geometric** condition.

Theorem 1 (Bracci-G.-Zimmer, 2018)

Let $D \subset \mathbb{C}^d$, D convex. If D admits a complete Kähler metric g, with $-B \leq Bisec(g) \leq -A < 0$ outside $K \subset\subset D$, then :

1. *D does not contain any non trivial complex line,*
2. *∂D does not contain any non trivial analytic disk,*
3. *$\partial D \in \mathcal{C}^\infty \Rightarrow \partial D$ has $< \infty$ type in the sense of d'Angelo.*

Theorem 1 (Bracci-G.-Zimmer, 2018)

Let $D \subset \mathbb{C}^d$, D convex. If D admits a complete Kähler metric g, with $-B \leq Bisec(g) \leq -A < 0$ outside $K \subset\subset D$, then :

1. *D does not contain any non trivial complex line,*
2. *∂D does not contain any non trivial analytic disk,*
3. *$\partial D \in \mathcal{C}^\infty \Rightarrow \partial D$ has $< \infty$ type in the sense of d'Angelo.*

Conjecture. Let $D \subset\subset \mathbb{C}^d$, D pseudoconvex, $\partial D \in \mathcal{C}^\infty$. Then D admits a complete Kähler metric g, with $-B \leq Bisec(g) \leq -A < 0$ outside $K \subset\subset D$ iff. D is of finite type.

Question 2. Which complex manifolds admit a complete Kähler metric with constant hol. sect. curvature outside a compact set ?

Question 2. Which complex manifolds admit a complete Kähler metric with constant hol. sect. curvature outside a compact set ?

Theorem 2 (G.-Zimmer, 2022)

Suppose that X is a Stein manifold with $\dim_{\mathbb{C}} X \geq 2$, $K \subset X$ is a compact subset where $X \setminus K$ is connected, and g_0 is a Hermitian metric on $X \setminus K$ which is complete at infinity. If g_0 is locally symmetric, then there exists a complete locally symmetric Hermitian metric g on X such that $g = g_0$ on $X \setminus K$.

Pf. Thm.2

Question 2. Which complex manifolds admit a complete Kähler metric with constant hol. sect. curvature outside a compact set ?

Theorem 2 (G.-Zimmer, 2022)

Suppose that X is a Stein manifold with $\dim_{\mathbb{C}} X \geq 2$, $K \subset X$ is a compact subset where $X \setminus K$ is connected, and g_0 is a Hermitian metric on $X \setminus K$ which is complete at infinity. If g_0 is locally symmetric, then there exists a complete locally symmetric Hermitian metric g on X such that $g = g_0$ on $X \setminus K$.

Pf. Thm.2

Since Kähler metric with cst. hol. sect. curv. is loc. symmetric:

Corollary 1 (G.-Zimmer, 2022)

Suppose that X is a Stein manifold with $\dim_{\mathbb{C}} X \geq 2$ and g_0 is a complete Kähler metric on X. If there exists a compact set $K \subset X$ such that g_0 has constant negative holomorphic sectional curvature on $X \setminus K$, then the universal cover of X is biholomorphic to the unit ball in $\mathbb{C}^{\dim_{\mathbb{C}} X}$.

Proof of Corollary 1.

Step 1. g_0 is loc. symmetric on $X \setminus K$.

Let $z \in X \setminus K$ and let $B_{g_0}(z, r)$ normal ball contained in $X \setminus K$ ($r << 1$).
Let $s_z := \exp_z \circ(-id_{T_z X}) \circ \exp_z^{-1} : B(z, r) \to B(z, r)$.
Then s_z is a holomorphic local isometry, i.e. $(X \setminus K, g_0)$ Hermitian loc. symmetric space.

Step 2. By Thm. 2, $g_0 = g_{|(X \setminus K)}$ where g complete locally symmetric Hermitian metric on X. Thm.3

The universal cover of (X, g) is a Hermitian symmetric space, hence has a transitive group.

Ccl. $H(g) = -c$ on X : $\tilde{X} \simeq \mathbb{B}^{dim_{\mathbb{C}} X}$ (Hawley-Igusa).

Application. Characterize bounded domains in $\mathbb{C}^d$ for which the Kobayashi metric is a Kähler metric.

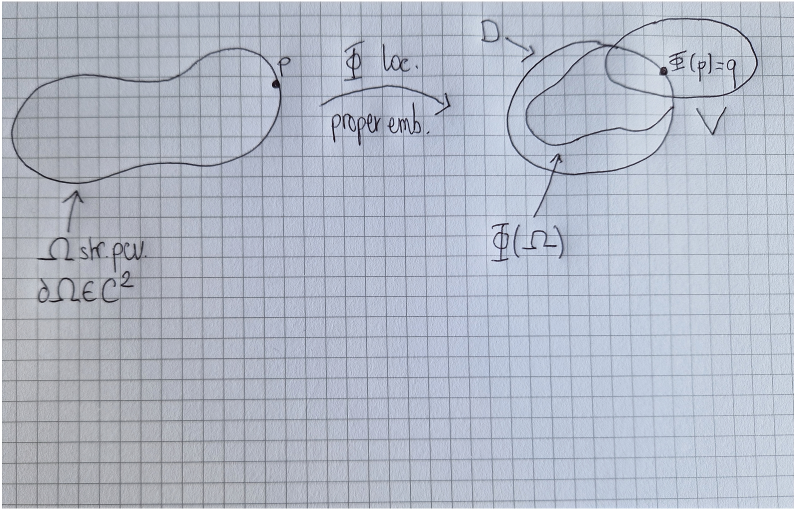
Theorem 3 (G.-Zimmer, 2022)

Suppose that $\Omega \subset \mathbb{C}^d$ is a bounded strongly pseudoconvex domain with $\mathcal{C}^2$ boundary. Then the following are equivalent:

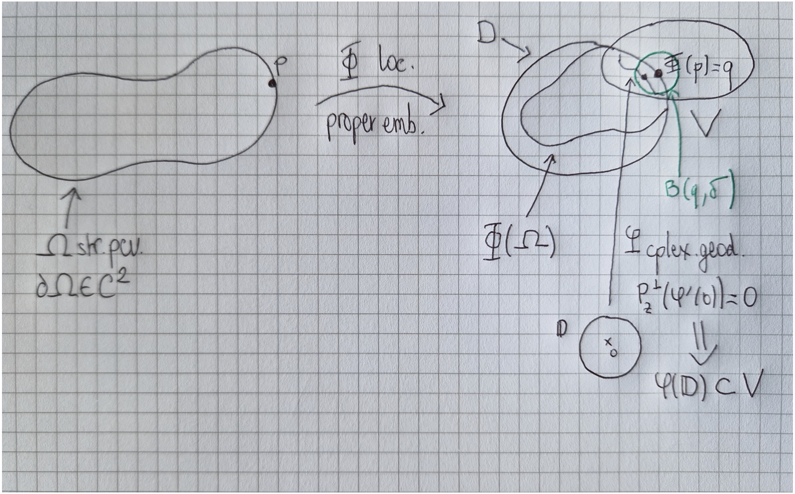
1. *the Kobayashi metric on Ω is a Kähler metric,*
2. *the Kobayashi metric on Ω is a Kähler metric with constant holomorphic sectional curvature,*
3. *the universal cover of Ω is biholomorphic to the unit ball.*

Rk. What if Ω is a noncompact complete hyperbolic manifold ?

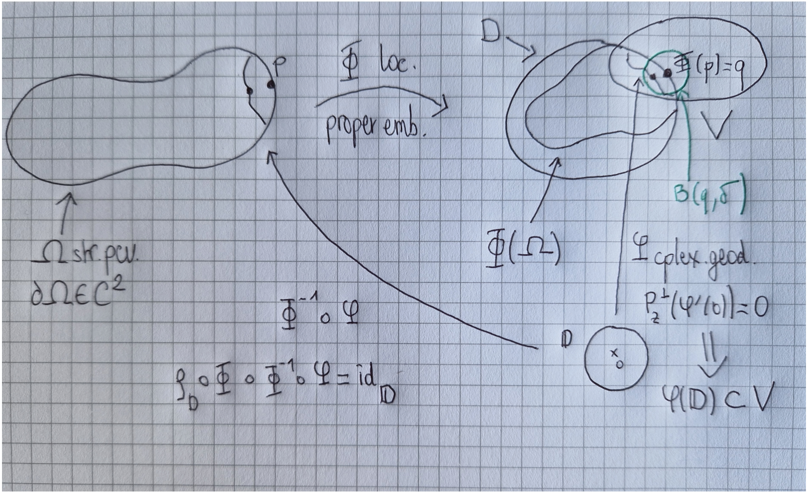
Step 1. Locally proper holoporphic embedding $\Phi : \Omega \to D$ str. convex



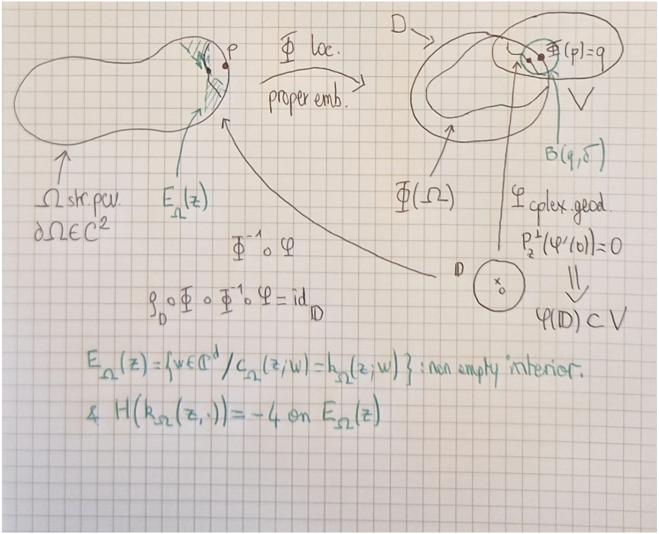
Step 2. Uniform behaviour of tangential complex geodesics in D



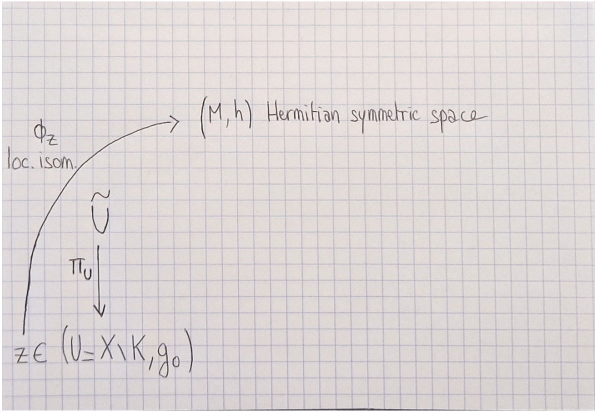
Step 3. "Almost tangential" small complex geodesics in Ω are isometries from $(\mathbb{D}, k_{\mathbb{D}})$ to (Ω, k_{Ω})



Step 4. $H(k_\Omega(z,\cdot)) = -4$ for z near $\partial\Omega$

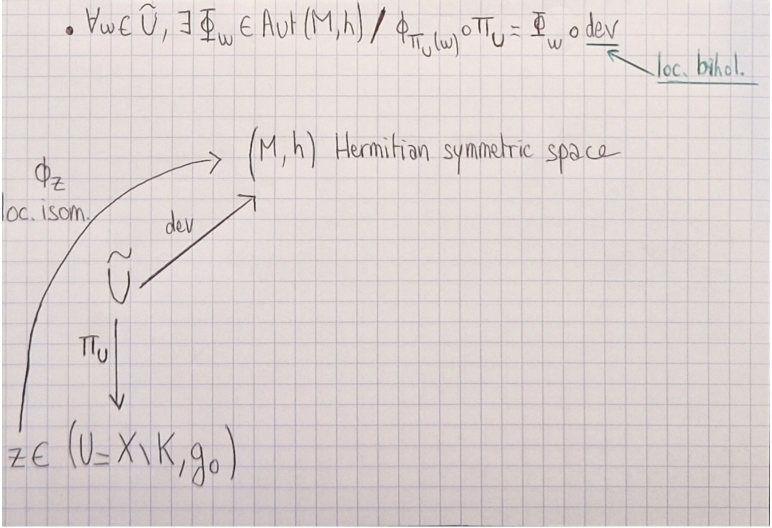


Proof of Theorem 2 Thm.2

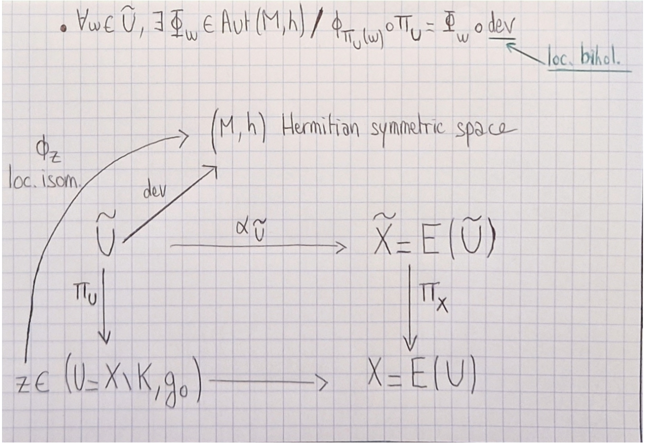


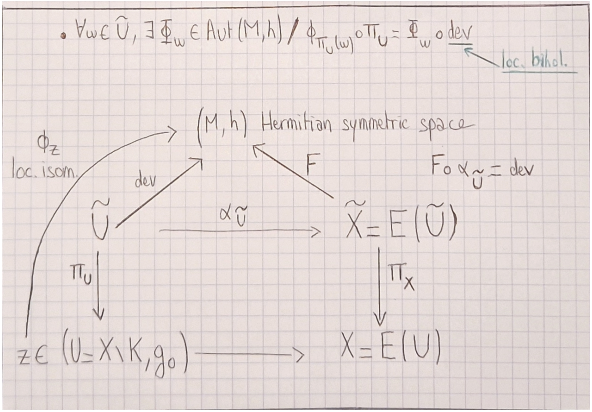
Step 1. There exists (M, h) simply connected, Hermitian symmetric space s.t. : $\forall z \in U,\ \exists\, O_z \subset U, \exists\, \phi_z : O_z \overset{\text{loc. isom.}}{\longrightarrow} M$.

- M is Stein and has no compact factor.


• ∀w ∈ Ũ, ∃ Φ_w ∈ Aut(M, h) / φ_{π_U(w)} ∘ π_U = Φ_w ∘ dev
loc. bihol.
(M, h) Hermitian symmetric space
φ_z
loc. isom.
dev
Ũ
π_U
z ∈ (U = X \ K, g_0)

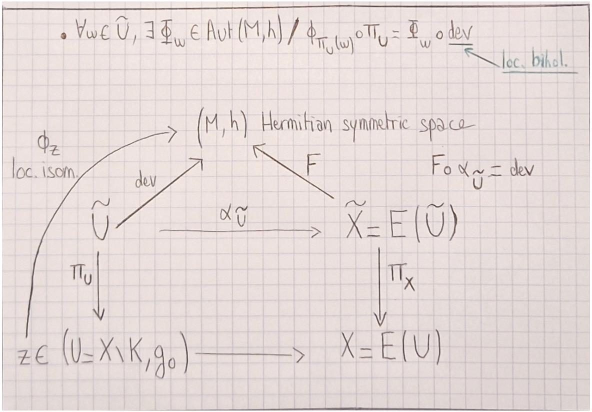
Step 2. Kerner's Theorem : $\tilde{X} = E(\tilde{U})$





Lemma 1

Let dev : $\tilde{U} \to M$ *developing map.*
(U is Stein, M is Stein) $\Rightarrow \exists F : \tilde{X} \overset{\text{loc. bihol.}}{\longrightarrow} M / \; F \circ \alpha_{\tilde{U}} = dev$.



Step 3. The Kähler metric F^*h descends to a loc. symmetric g on X and $g_{|U} = g_0$. Finally g is complete on X. □