Sharp subelliptic estimates for the $\bar{\partial}$-Neumann problem

joint w. S. Mongodi (Univ. Milano Bicocca)

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Let Ω be a domain in $\mathbb{C}^n$. The celebrated $\overline{\partial}$-**problem** (a.k.a. the inhomogenous C-R eq.'s) is the system of first order PDEs

$$\begin{cases} \bar{\partial} f = u \\ \bar{\partial} u = 0 \end{cases}$$

where f is a complex-valued function and u is a $(0,1)$-form, both defined on Ω.

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In this talk I want to focus on the **local regularity problem** for $\bar{\partial}$.

- **(Interior) ellipticity**:

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- Lack of uniqueness: f is a sol. iff $f + g$ is a sol., for any holomorphic g $\Longrightarrow$ to talk about **regularity at the boundary** we need to choose a **particular solution**.
- (Spencer's extension of) **Hodge theory** provides the right framework to do that, leading to the $\overline{\partial}$-**Neumann problem**, a noncoercive boundary value problem to which the elliptic theory of BVPs cannot be applied.

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L^2 Existence Theorem (Hörmander 1965)

If the domain Ω is bounded and **pseudoconvex**, then the $\overline{\partial}$-problem

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The **canonical solution** is the one orthogonal to the null-space of $\overline{\partial}$ in $L^2(\Omega)$, a.k.a. the Bergman space $A^2(\Omega)$.

The L^2 existence theorem boils down to the **global a priori estimate**

$$\underbrace{||\overline{\partial}u||^2 + ||\overline{\partial}^* u||^2}_{\text{quad. form of complex Laplacian}} \gtrsim ||u||^2 \qquad \forall u \in \mathcal{D}_{0,1} = C^\infty_{0,1}(\overline{\Omega}) \cap \mathrm{dom}(\overline{\partial}^*)$$

where $||\cdot||$ are L^2 norms, and

$$\mathcal{D}_{0,1} = \{u \in C^\infty_{0,1}(\overline{\Omega}) \colon\ \mathrm{int}(\overline{\partial}r)u = 0 \text{ on } b\Omega\} \qquad (r \text{ def. fct. of } \Omega).$$

Subellipticity Problem

Assume now that Ω is bdd pscvx and **smooth**. Is the $\overline{\partial}$-problem

$$\begin{cases} \overline{\partial} f = u \in L^2_{0,1}(\Omega) \\ \overline{\partial} u = 0 \\ f \perp \mathcal{O}(\Omega) \cap L^2(\Omega) \end{cases}$$

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More precisely, given $x_0 \in b\Omega$, is there an $s > 0$ with the property that (for every $t \geq 0$)

$$u \in H^t_{(0,1)} \text{ near } x_0 \quad \Longrightarrow \quad f \in H^{t+s} \text{ near } x_0 \text{ ?}$$

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In this case, one says that the problem is s-**subelliptic** at the boundary point x_0.

It is enough to prove the subelliptic gain of regularity at the level $t = 0$.

Theorem (Kohn–Nirenberg 1965)

Let Ω be bdd smooth pscvx and let $x_0 \in b\Omega$. Assume that the a priori **subelliptic estimate of order** $s > 0$

$$||\overline{\partial} u||^2 + ||\overline{\partial}^* u||^2 \gtrsim ||u||_s^2 \qquad \forall u \in \mathcal{D}_{0,1} : \ \mathrm{supp} u \subseteq U$$

holds, where $|| \cdot ||_s$ is the H^s Sobolev norm, and U is a neighborhood of x_0.

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holds, where $|| \cdot ||_s$ is the H^s Sobolev norm, and U is a neighborhood of x_0. Then the $\overline{\partial}$-problem on Ω is s-subelliptic at x_0:

$$\text{if the datum } u \in H^t_{(0,1)} \text{ near } x_0 \quad \Longrightarrow \quad \text{canonical sol. } f \in H^{t+s} \text{ near } x_0$$

Subellipticity problem

It comes in three flavors:

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2. **Quantitative**: determine a **quantitatively controlled** s s.t. an s-subelliptic estimate holds/does not hold at a given boundary point x_0.
3. **Sharp**: determine the **maximal* value** $s = s_{\mathrm{sharp}}(\Omega; x_0)$ such that an s-subelliptic estimate holds at a given boundary point x_0.

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The answers to these problems are deeply tied to the local CR geometry of the boundary near the point x_0 of interest.

A lot is known about (1) and (2), but (3) is widely open, and very (unreasonably?) difficult in general (Catlin–D'Angelo 2010).

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2. the D'Angelo type $x_0 \mapsto \Delta^1(b\Omega; x_0)$ is not upper semicontinuous (D'Angelo 1982).

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Upper semicontinuous envelope of the type? More on this later.

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3. **Kohn 1979**: Subelliptic multiplier ideals: if $b\Omega$ is real-analytic near x_0, then $s_{\mathrm{sharp}}(\Omega; x_0) > 0$ iff $b\Omega$ contains no (possibly singular) complex curve through x_0.

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7. **Zimmer 2022**: $s_{\rm sharp}(\Omega; x_0) \geq 1/\max \text{ line type}$ on convex finite type domains (via Gromov hyperbolicity).

In the rest of the talk I am going to:

1. present an approach to subellipticity on rigid domains

$$\Omega = \{(z, z_{n+1}) \colon \operatorname{Im}(z_{n+1}) > \varphi(z)\}, \quad \varphi \quad \text{plush},$$

   alternative to Catlin's potential-theoretic method and Kohn's algorithm(s), and based on **spectral gap estimates** for an appropriate energy form E^{φ}.

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Let's begin by defining homogeneous rigid domains.

Dfn

A domain $\Omega \subset \mathbb{C}^{n+1}$ is said to be a d-**homogeneous special domain** if

$$\Omega = \left\{ (z, z_{n+1}) \in \mathbb{C}^{n+1} \colon \operatorname{Im}(z_{n+1}) > \sum_{k=1}^{n} |F_k(z)|^2 \right\},$$

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where each F_k is a homogeneous polynomial of degree d.
Such a domain is of D'Angelo finite type (everywhere) iff

$$\{F_1 = \ldots = F_n = 0\} = \{0\},$$

that is, iff $[z_1 : \cdots : z_n] \longmapsto [F_1(z_1, \ldots, z_n) : \cdots : F_n(z_1, \ldots, z_n)]$ is a globally defined holomorphic self-map of the projective space

$$\mathrm{F} : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$$

Homogeneous special domains have various attractive features:

1. the CR geometry of their boundary (in particular D'Angelo and other "types") boils down to the structure of the singularities of $F : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$, that is, critical points, rank of the Jacobian at critical points, geometry of the null space of the Jacobian, ...

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2. two symmetries to spend: homogeneity and rigidity (translation invariance in $\mathrm{Re}(z_{n+1})$).
3. rich enough to display a variety of behaviors, including the aforementioned lack of semicontinuity of the type. E.g.,

$$\Omega = \{\mathrm{Im}(z_4) > |z_1^2 + z_2 z_3|^2 + |z_2^2 + z_1 z_3|^2 + |z_3^2|^2\}$$

has $\Delta^1(b\Omega; 0) = 4$ and $\Delta^1(b\Omega; p) = 8$ on a 3-dim submanifold accumulating at 0.

The key objects in our analysis are the

Energy forms E^φ

Given φ plush we define

$$\mathsf{E}^\varphi(u) = \int_{\mathbb{C}^n} |\nabla^{0,1} u|^2 e^{-2\varphi} + \int_{\mathbb{C}^n} \lambda_1 |u|^2 e^{-2\varphi} \qquad (u \in C^\infty_c(\mathbb{C}^n)),$$

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Antecedents: **Christ 1991, Berndtsson 1996, Haslinger–Helffer 2007 etc.**

The relevance of the energy forms E^φ to the subellipticity problem is revealed by the following

Lemma 1. Spectral gap estimates imply subellipticity

Let φ be $2d$-homogeneous plush, and let Ω be the associated rigid domain. Assume that

$$\mathrm{E}^\varphi(\psi) \gtrsim R^{-2+4ds} \int_{\mathbb{C}^n} |\psi|^2 e^{-2\varphi} \qquad \forall \psi \in C_c^\infty(B(R)), \quad \forall R \text{ large.}$$

Then $s_{\mathrm{sharp}}(\Omega; 0) \geq s$.

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Proof uses reduction to a $\bar{\partial}_b$ problem, Fourier analysis in the rigid direction, homogeneity, and a bit of microlocal analysis. All ingredients are standard.

Detour: Spectral gaps for Schrödinger operators imply subellipticity for Grushin operators

Let $V : \mathbb{R}^n \to [0, +\infty)$ be a potential and $\hbar > 0$. We are interested in the bottom of the spectrum of $-\hbar^2\Delta + V(x)$, i.e.,

$$b_V(\hbar) = \inf_{\int_{\mathbb{R}^n} |\psi(x)|^2 \, dx = 1} \left\{ \underbrace{\hbar^2 \int_{\mathbb{R}^n} |\nabla \psi(x)|^2 \, dx}_{\text{kinetic en.}} + \underbrace{\int_{\mathbb{R}^n} V(x) |\psi(x)|^2 \, dx}_{\text{potential en.}} \right\}$$

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Model: $V(x) \simeq |x|^p$ ($p > 0$ is the "type"). By the **uncertainty principle**

$$\text{“}\delta x \cdot \delta p \gtrsim \hbar\text{”} \qquad (p = -i\hbar\nabla)$$

the total energy of a "wave-packet" localized on $B(0, R)$ is about

$$\frac{\hbar^2}{R^2} + R^p \quad \Longrightarrow \quad \min \simeq \hbar^{\frac{2p}{p+2}}.$$

This yields sharp sub-elliptic estimates for the degenerate elliptic operator $\mathcal{L} = -\Delta - V(x)\frac{\partial^2}{\partial t^2}$ on $\mathbb{R}^{n+1}$.

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Taking the Fourier transform in t, we get the one-parameter family of PDEs

$$\widehat{\mathcal{L}}(\xi) = -\Delta + \xi^2 V(x) = \xi^2\left(-\hbar^2\Delta + V(x)\right) \qquad (\hbar = |\xi|^{-1}).$$

For the purposes of regularity theory, the limit $|\xi| \to +\infty$ is of interest. This coincides with the **semiclassical limit** $\hbar \to 0$:

$$\text{bottom of spectrum of } \widehat{\mathcal{L}}(\xi) \simeq |\xi|^{\frac{4}{p+2}}$$
$$\Longrightarrow \quad (\mathcal{L}f, f)_{L^2(\mathbb{R}^{n+1})} \gtrsim \|f\|^2_{W^{\frac{2}{p+2},2}},$$

Theorem (Grushin)

The operator $\mathcal{L}$ is sub-elliptic of order $\frac{2}{p+2}$ (which is essentially the reciprocal of the type) at points of the degeneracy locus $x = 0$.

Inspired by the above we look with new eyes at

$$\mathsf{E}^{\varphi}(u) = \underbrace{\int_{\mathbb{C}^n} |\nabla^{0,1} u|^2 e^{-2\varphi}}_{\bar{\partial}-\text{kinetic en.}} + \underbrace{\int_{\mathbb{C}^n} \lambda_1 |u|^2 e^{-2\varphi}}_{\text{potential en.}}$$

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The enemies are "wavepackets" u that are **sharply localized near the degeneration set**

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and have small $\bar{\partial}$-kinetic energy, that is, are "almost holomorphic".

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Is there an appropriate "uncertainty principle" ruling out their existence?

Lemma 2. One-dimensional Uncertainty Principle for the $\frac{\partial}{\partial \bar{z}}$ Operator

Let $\varphi : \mathbb{D} \to \mathbb{R}$ be subharmonic.
Assume that $\Delta\varphi$ **is a perturbation of** $|Az^d|^2$ $(A \in \mathbb{C})$. Then

$$\int_{\mathbb{D}} \left|\frac{\partial u}{\partial \bar{z}}\right|^2 e^{-2\varphi} + \int_{\mathbb{D}} \Delta\varphi |u|^2 e^{-2\varphi} \gtrsim |A|^{\frac{2}{d+1}} \int_{\mathbb{D}} |u|^2 e^{-2\varphi}$$

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- The perturbation is quite delicate: need to control clustering of zeros (harmonic analysis techniques).

The second key ingredient in our approach to spectral gap estimates for E^{φ} is the following notion.

Approximate minimal eigenvector fields

Let $\varphi : \mathbb{C}^n \to \mathbb{R}$ be plush. An **approx. minimal eigenvector field for** φ **at** $p \in \mathbb{C}^n$ is a germ at p of holomorphic vector field $X(z)$ with the properties that $X(p) \neq 0$ and

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} X_j(z)\overline{X_k(z)} \lesssim \lambda_1(z)|X(z)|^2.$$

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Lemma 3. Existence of approx. minim. e. v.-f.'s for co-rank 1

If $\varphi(z) = \sum_{k=1}^{n} |F_k(z)|^2$ with F_k's holomorphic, then approx. minim. e. v.-f.'s exist **at every point where the Levi form of φ has co-rank** 1.

Lemma 2 and Lemma 3 combined provide lower bounds on E^{φ}, **for appropriate** φ**'s**.

The strategy is the following:

- Cut $\mathbb{C}^n$ into dyadic shells $\{z \in \mathbb{C}^n \colon \lambda^m \leq |z| \leq \lambda^{m+1}\}$, where $\lambda > 1$ is appropriately chosen and $m \geq 1$ (local contribution is easy to deal with).

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- Apply one-dimensional uncertainty principle to each disc (Lemma 2).

The strategy above allows to prove the following

Theorem

Let $\Omega \subset \mathbb{C}^{n+1}$ be d-homogeneous special of finite type (at the origin), with associated map $\mathrm{F} : \mathrm{P}^{n-1} \to \mathrm{P}^{n-1}$. **Assume that co-rank of Jacobian of F is at most one***.

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$$s_{\mathrm{sharp}}(\Omega; 0) = \frac{1}{2\max\{d, \mathfrak{t}(\mathsf{F})\}},$$

where $\mathfrak{t}(\mathsf{F}) = \sup_\gamma \nu(\mathsf{F} \circ \gamma)$, sup is over non-singular analytic discs $\gamma : \mathbb{D} \to \mathsf{P}^{n-1}$ and $\nu(G)$ is the order of vanishing of $G - G(0)$.

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The quantity $2\max\{d, \mathfrak{t}(\mathsf{F})\}$ equals the upper semicontinuous envelop of the D'Angelo type at the origin.

*This forces low-dimensionality: $n + 1 \leq 5$.

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Thanks a lot for your attention!