Oka properties of complements of closed convex sets in $\mathbb{C}^n$

Joint work with Franc Forstnerič

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Oka manifolds

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Definition

A complex manifold X is said to be an Oka-manifold if for any convex set $K \subset \mathbb{C}^n$ and any holomorphic map $f : K \to X$, there exist entire maps $F : \mathbb{C}^n \to X$ approximating f on K.

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Hence, a key object in Oka theory is to "find as many Oka manifolds as possible".

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Example

Consider

$$E = \{z = (z_1, ..., z_n) : \mathfrak{Im}(z_n) \geq 0\}$$

Then $\mathbb{C}^n \setminus E$ is certainly not an Oka manifold, since $\mathbb{C}^n \setminus E$ is biholomorphic to a product $H \times \mathbb{C}^{n-1}$, where H is a 1-dimensional halfplane.

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Theorem

/corollary Let $\phi(z_1, ...z_{n-1}, \mathfrak{Re}(z_n))$ *be a real valued strictly convex function. Then*

$$\Omega = \{z \in \mathbb{C}^n : \mathfrak{Im}(z_n) < \phi(z_1, ...z_{n-1}, \mathfrak{Re}(z_n))\}$$

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Theorem

(A)
Suppose that $E \subset \mathbb{C}^n, n \geq 2$, is a closed domain with $\mathcal{C}^1$-smooth boundary, and assume that no $T_p^{\mathbb{C}} bE$ does contains a real half line. Then $\mathbb{C}^n \setminus E$ is an Oka-manifold.

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Theorem

(B) Suppose that $E \subset \mathbb{C}^n, n \geq 2$ is a closed domain with $\mathcal{C}^1$-smooth boundary which is strongly convex. Then $\mathbb{C}^n \setminus E$ is an Oka manifold.

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Theorem

(B) Suppose that $E \subset \mathbb{C}^n, n \geq 2$ *is a closed domain with* $\mathcal{C}^1$*-smooth boundary which is strongly convex. Then* $\mathbb{C}^n \setminus E$ *is an Oka manifold.*

Theorem

(C) Suppose that $E \subset \mathbb{C}^n$ *is a closed convex set that does not contain a real line. Then* $\mathbb{C}^n \setminus E$ *is an Oka manifold.*

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Theorem

(M) Suppose that $E \subset \mathbb{C}^n$ is a closed set such that there exists a hyperplane $\Lambda \in \mathbb{CP}^n$ such that $\bar{E}$ is polynomially convex in $\mathbb{CP}^n \setminus \Lambda$. Then $\mathbb{C}^n \setminus E$ is an Oka-manifold.

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To prove this results we need two remarkable results of Kusakabe.

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Theorem

(Kusakabe, Ell_1-characterization) Let X^n be a complex manifold. Assume that for any compact convex set $K \subset \mathbb{C}^m$ and any holomorphic map $f : K \to X$ there exists a holomorphic map

$$F : K_z \times \mathbb{C}^N_\zeta \to X$$

such that $F(z, 0) = f(z)$ and $\frac{\partial F}{\partial \zeta}(z, 0)$ has rank n. Then X is an Oka manifold.

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Theorem

(Kusakabe, localization) Let X be a complex manifold, and suppose that $X = \cup_{j=1}^k \Omega_j$ where Ω_j is Zariski open and Oka. Then X is Oka.

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- Use Kusakabe's result and Andersén-Lempert Theory to show that $\mathbb{C}^n \setminus (E \cup \Lambda)$ is an Oka manifold.
- Do the same for finitely many hyperplanes $\Lambda_j, j = 1, ..., k$ such that $\bigcap_j \Lambda_j = \emptyset$, and use the localization theorem to conclude that $\mathbb{C}^n \setminus E$ is Oka.

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Hence the "main theorem" applies to projectively convex sets E.

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- Let $p \in \mathbb{CP}^n \setminus \bar{E}$ and chose a complex line L passing through p. One can slide L until it is a tangent to ∂E at some point.
- Then boundary ∂E is connected.

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- Let $E \subset \mathbb{C}^n$ be a Stein compact. Is $\mathbb{C}^n \setminus E$ an Oka manifold?
- Let E be the closure of a bounded, smoothly bounded, pseudoconvex domain. Is $\mathbb{C}^n \setminus E$ an Oka manifold?

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- Let E be the closure of a bounded, smoothly bounded, pseudoconvex domain. Is $\mathbb{C}^n \setminus E$ an Oka manifold?
- Is the complement of the Diederich-Fornæss-Worm an Oka manifold?