Monge-Ampère energies

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Fix (X, ω) a compact Kähler manifold, $\dim_{\mathbb{C}} X = n \geq 1$.

- ω is a Kähler form (a $(1,1)$-form real, closed, positive)
- normalized s.t $\int_X \omega^n = 1$.

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Disclaimer: I will work with ω as reference form but everything still holds when working with a merely big form θ.

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$$\boxed{\int_X \omega_\varphi^n \leq \int_X \omega^n = 1}$$

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A: YES

Theorem (Darvas, Di Nezza, Lu '21)

Assume μ is a positive non-pluripolar measure s.t. $\mu(X) = m \in (0,1]$. Then $\exists! \omega$-psh function $\varphi \in \mathcal{E}(X, \omega, \phi)$ with $\sup_X \varphi = 0$ s.t. $\omega_\varphi^n = \mu$.

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Ex: $\phi = 0$

ϕ with analytic singularites, i.e. $\phi \sim \log \sum_k |f_k|^2$+smooth, f_k holo

The energy class (of relative full mass potentials) is then defined as

$$\mathcal{E}(X,\omega,\phi) := \{u \in \mathrm{PSH}(X,\omega) \ : \ u \le \phi + C \ \text{ and } \ \int_X \omega_u^n = \int_X \omega_\phi^n\}$$

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RK1: $u \sim \phi$ (i.e. $|u-\phi| \le C$) $\Longrightarrow \int_X \omega_u^n = \int_X \omega_\phi^n$. [Witt-Nyström '17]

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Ex: bounded functs, $-(-\log|z|)^\alpha \in \mathcal{E}$
BUT $\log|z| \notin \mathcal{E}$

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→ I am going to define weighted subspaces of $\mathcal{E}(X, \omega, \phi)$.

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In [DDL20] we then defined the relative Monge-Ampère χ-energy class

$$\mathcal{E}_\chi(X,\omega,\phi) := \{u \in \mathcal{E}(X,\omega,\phi) \ : \int_X \chi(|u-\phi|)\,\omega_u^n < +\infty\}$$

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- [$\phi = 0$] [EGZ09], [BEGZ10], [BBEGZ13]: $\exists$KE metrics on singular variety
- [ϕ *model*] [DDL20/21]: $\exists$ singular KE metrics *(with prescribed singularities)*

Observe that being in an energy class gives some information on "how fast" a ω-psh function goes to $-\infty$

Ex: Take $\phi = 0$, $\chi(t) = t^p$, $p > 0$. Then

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FACT: $\{u \in \mathrm{PSH}(X,\omega) \ : \ |u-\phi| \leq C, C > 0\} = \cap_\chi \mathcal{E}_\chi(X,\omega,\phi)$.

$$[\phi = 0] \qquad \mathrm{PSH}(X,\omega) \cap L^\infty(X) = \cap_\chi \mathcal{E}_\chi(X,\omega).$$

Theorem (Darvas, Di Nezza, Lu '23)

Assume μ is a positive non-pluripolar measure s.t. $\mu(X) = m \in (0,1]$. Then TFAE:

- $\exists ! \varphi \in \mathcal{E}_\chi(X,\omega,\phi)$ *with* $\sup_X \varphi = 0$ *s.t.* $\omega_\varphi^n = \mu$
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RK:

- $\phi = 0$, $\chi(t) = t^p$: Guedj-Zeriahi '05 and they asked the same question for more general weights.
- $\phi = 0$, χ with "growth condition": Thai-Vu '21

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- Assumption: $\int_X \chi(|u|)\,\omega_\varphi^n \leq A \int_X \chi(|u|)\,\omega_v^n$
- $v \in L^\infty \leftrightarrow v \in E_\chi$ $\forall \chi$. DDL $\Longrightarrow \int_X \chi(|u|)\,\omega_v^n < +\infty$ $\forall u \in E_\chi$

Best evidence we can get:

Theorem (Darvas, Di Nezza, Lu 23)

Assume $\varphi \in \mathcal{E}(X,\omega)$ *s.t.* $\omega_\varphi^n \leq A\omega_v^n$, $A > 0$, $v \in L^\infty(X)$. *Then* $\exists \alpha > 0$ *s.t.*

$$\int_X e^{\alpha|\varphi|}\omega_\varphi^n < +\infty.$$

RK1: We also have a relative version of it.