Bergman functions associated to measures on totally real submanifolds

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Introduction

• In various problems from approximation theory or orthogonal polynomials, it is crucial to understand the asymptotic behaviour of the Bergman kernel function (or equivalently its inverse known as Christoffel's function).

• Even more generally, one also needs to study the asymptotic behaviour of the Bergman kernel (or Christoffel-Darboux kernel).

• Distribution of Fekete's points, Sampling or interpolation of polynomials (approximation theory). Chrifstoffel's function is a classical and main topic in orthogonal polynomials with various connections to other fields (e.g., random matrices, zeros of random polynomials).

• Long history: at least as old as Bergman kernel in complex geometry (Szegő's book "Orthogonal polynomials" 1939).

Orthogonal polynomials

• Let K be a non-pluripolar compact subset in $\mathbb{C}^n$ (e.g., any Jordan arc in $\mathbb{C}$, any bounded open subset in $\mathbb{R}^n \subset \mathbb{C}^n$).

•. Let μ be a probability measure whose support is non-pluripolar and is contained in K, and Q be a real continuous function on K.

• Most standard settings are measures supported on concrete domains on $\mathbb{R}^n \subset \mathbb{C}^n$ (such as balls or simplexes in $\mathbb{R}^n$) or in the unit ball in $\mathbb{C}^n$. Measures supported on finite unions of piecewise smooth Jordan curves $\mathbb{C}$ or domains in $\mathbb{C}$ bounded by Jordan curves.

• Considering measures on $\mathbb{C}^n$ whose support are not necessarily in $\mathbb{R}^n$ are also important in many applications (among other things, Lasserre-Pauwels-Putinar's book: The Christoffel–Darboux Kernel for Data Analysis).

Bergman's kernel function

• Let K be a non-pluripolar compact subset in $\mathbb{C}^n$. Let μ be a probability measure whose support is non-pluripolar and is contained in K, and Q be a real continuous function on K.
• Let $\mathcal{P}_k$ be the space of restrictions to K of complex polynomials of degree at most k on $\mathbb{C}^n$.
• Let $(s_1, \ldots, s_{d_k})$ be an orthonormal basis of $\mathcal{P}_k$ with respect to the $L^2(\mu, kQ)$-norm. The Bergman kernel function of order k associated to μ with weight Q is

$$B_k(x) := \sum_{j=1}^{d_k} |s_j(x)|^2 e^{-2kQ(x)} = \sup_{s \in \mathcal{P}_k} |s(x)|^2 e^{-2kQ(x)} / \|s\|^2_{L^2(\mu, kQ)}.$$

When $Q \equiv 0$, we say that B_k is *unweighted*. In this case ($Q \equiv 0$) the inverse of B_k is known as the Christoffel function in the literature on orthogonal polynomials.

- Consider the projection $\rho : L^2(\mu, kQ) \to \mathcal{P}_k$. The Bergman kernel of order k associated to (K, μ, Q) is the Schwartz kernel of ρ:

$$\mathbf{K}_k(x,y) := \sum_{j=1}^{d_k} s_j(x)\overline{s_j(y)}.$$

The norm of $\mathbf{K}_k$ with respect to the weight Q is

$$|\mathbf{K}_k(x,y)|_Q := \left| \sum_{j=1}^{d_k} s_j(x)e^{-kQ(x)}\overline{s_j(y)}e^{-kQ(y)} \right|.$$

One has

$$B_k(x) = |\mathbf{K}_k(x,x)|_Q.$$

Analogous construction if $\mathbb{C}^n \subset \mathbb{P}^n$ is replaced by a compact Kähler manifold X, and polynomials replaced by holomorphic sections of a positive line bundle on X.

Bergman's kernel

Fundamental problem: Study asymptotic properties of Bergman kernel function or more generally Bergman kernel.

• The question has been extensively studied in dimension one. Very few in higher dimension: some particular cases (Kroo, Lubinsky, Totik, etc.), a general approach via pluripotential theory.

• When $K = X$ (a compact polarised manifold), μ a smooth volume form, Bergman kernel is an object of intensive study in complex geometry (motivations are different from ours): Tian, Catlin, Zelditch, etc; also Ma-Marinescu's book. In this case

$$B_k(x) = c_n k^n + O(k^{n-1})$$

• Crucial difficulty in our general setting: global analysis on manifolds don't apply to the setting of our problem. Alternative (or additional) main tool is Pluripotential theory.

Plurisubharmonic envelope

• Asymptotic behaviour of Bergman kernel functions is closely related to the notions of the extremal plurisubharmonic envelope and the equilibrium measure associated to (K, Q).

• Let

$$V_{K,Q} := \sup\{\psi \in \mathcal{L}(\mathbb{C}^n) : \psi \leq Q \quad \text{on } K\},$$

where $\mathcal{L}(\mathbb{C}^n)$ is the Lelong class of psh functions on $\mathbb{C}^n$ such that $\psi - \log(\|z\| + 1)$ is bounded from above on $\mathbb{C}^n$.

• Again: analogue in the setting of projective polarised manifold. When $K = X$: Intensively studied in the theory of complex Monge-Ampère equations.

• Let $V^*_{K,Q}$ be the upper semi-continuous regularisation of $V_{K,Q}$. Note $V^*_{K,Q}$ is locally bounded and belong to $\mathcal{L}(\mathbb{C}^n)$.

• Key question is the regularity of $V_{K,Q}$: the best is Hölder continuity in general.

Equilibrium measure

- Let $dd^c := i/\pi\partial\bar{\partial}$.

- The equilibrium measure associated to (K, Q) is the self-intersection $\mu_{K,Q} := (dd^c V^*_{K,Q})^n$ of $dd^c V^*_{K,Q}$.

- The measure $\mu_{K,Q}$ is supported on K and $V_{K,Q} = Q$ a.e. with respect to $\mu_{K,Q}$.

- Very hard to compute explicitly: if $K \subset \mathbb{R}^n$, $\mu_{K,Q}$ is equivalent to the Lebesgue measure on compact subsets in the interior of K. Explicit formula when $K \subset \mathbb{R}^n$ symmetric and of non-empty interior (Bedford-Taylor, also Baran, Levenberg, etc.).

- Let $K = [-1, 1]$, $Q \equiv 0$:
$V_{K,Q}(z) = \log|z + \sqrt{z^2 - 1}| \in \mathcal{C}^{1/2}(\mathbb{C})\backslash\mathcal{C}^{1/2+\epsilon}(\mathbb{C})$ for every $\epsilon > 0$, and

$$\mu_{K,Q} = \mathbf{1}_{[-1,1]}\frac{2dt}{\sqrt{1-t^2}}.$$

Bernstein-Markov measures

• Let $\mathcal{P}_k$ be the space of polynomials in $\mathbb{C}^n$ of degree at most k.

• The measure μ is said to be a Bernstein-Markov measure (with respect to (K, Q)) if for every $\epsilon > 0$, there exists $C > 0$ such that

$$\sup_K |s|^2 e^{-2kQ} \leq C e^{\epsilon k} \|s\|^2_{L^2(\mu, kQ)}$$

for every $s \in \mathcal{P}_k$.

• In other words, the Bergman kernel function of order k grows at most subexponentially (on K), *i.e.*,

$$\sup_K B_k = O(e^{\epsilon k})$$

as $k \to \infty$ for every $\epsilon > 0$.

• Examples of Bernstein-Markov measures and criteria checking this condition: (Bloom-Levenberg-Piazzon-Wielonsky, survey, 2015).

Generic Cauchy-Riemann submanifolds

• A smooth real submanifold Y in $\mathbb{C}^n$ is generic Cauchy-Riemann if for every $a \in Y$: $T_aY + JT_aY = \mathbb{C}^n$.

• Non-degenerate piecewise smooth generic Cauchy- Riemmann are non-pluripolar.

• A finite family of smooth Jordan arcs which are transverse to each other is a non-degenerate piecewise smooth generic Cauchy-Riemann submanifold in $\mathbb{C}$.

• Bounded domains with smooth boundary in $\mathbb{R}^n \subset \mathbb{C}^n$, compact maximally totally real submanifolds in $\mathbb{C}^n$ (any piecewise-smooth Jordan arc in $\mathbb{C}$).

• Classical examples: $K = [-1, 1]$ in $\mathbb{C}$ or $K = \mathbb{S}^1$ or $\overline{\mathbb{D}}$ in $\mathbb{C}$, more generally, K Polyhedron in $\mathbb{R}^n \subset \mathbb{C}^n$.

• It is important to consider K piecewise or having boundary.

Hölder regularity of K

Definition (Dinh-Ma-Nguyên, 2016)

For $\alpha \in (0,1]$ and $\alpha' \in (0,1]$, a non-pluripolar compact K is said to be $(\mathcal{C}^{0,\alpha}, \mathcal{C}^{0,\alpha'})$-regular if for any positive constant C, the set $\{V_{K,Q} : Q \in \mathcal{C}^{0,\alpha}(K) \text{ and } \|Q\|_{\mathcal{C}^{0,\alpha}(K)} \leq C\}$ is a bounded subset of $\mathcal{C}^{0,\alpha'}(\mathbb{C}^n)$.

The following provides examples for this regularity notion.

Theorem (V, 2018)

Let α be an arbitrary number in $(0,1)$. Then any compact generic nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold K of $\mathbb{C}^n$ is $(\mathcal{C}^{0,\alpha}, \mathcal{C}^{0,\alpha/2})$-regular. Moreover if K has no singularity, then K is $(\mathcal{C}^{0,\alpha}, \mathcal{C}^{0,\alpha})$-regular.

• The case where $K = X$ or K is a bounded open subset with smooth boundary in $\mathbb{C}^n$ (Dinh-Ma-Nguyên, 2016).

Hölder regularity of K

• Let K be a compact generic nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$.

Theorem (Marinescu-V, 2023 and V, 2018)

We have $V_K \in \mathcal{C}^{1/2}(\mathbb{C}^n)$, where $V_K := V_{K,Q}$ for $Q \equiv 0$.

• was conjectured by (Sadullaev-Zeriahi, 2016). Hölder regularity for V_K (for arbitrary K with uniform density in capacity) is a local property; (Nguyen, 2023).

Theorem (Marinescu-V, 2023, a Bernstein-Markov type inequality)

There exists a constant $C > 0$ such that for every complex polynomial p on $\mathbb{C}^n$ we have

$$\|\nabla p\|_{L^\infty(K)} \leq C(\deg p)^2 \|p\|_{L^\infty(K)}. \quad (1)$$

• known when K is algebraic in $\mathbb{R}^n$: Berman-Ortega-Cerda, Bos-Levenberg-Milman-Taylor, etc.

Bernstein-Markov measures

Theorem (Marinescu-V, 2023)

Let K be a compact generic Cauchy-Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Then K is locally regular. In particular $V_{K,Q} = V^*_{K,Q}$.

- It was known that K is locally regular if K is smooth real analytic (Berman-Boucksom-Witt Nyström).

Corollary (Marinescu-V, 2023)

Let K be a compact generic Cauchy-Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$. Let μ be a finite measure supported on K such that there exist constants $\tau > 0, r_0 > 0$ satisfying $\mu(\mathbb{B}(z,r) \cap K) \geq r^\tau$ for every $z \in K$, $r \leq r_0$. Then μ is a Bernstein-Markov measure with respect to (K, Q).

- Real analytic case and dimension 1 case was known.

• (Dinh-Nguyên, 2018) μ is said to be 1-Bernstein-Markov if for every constant $0 < \delta < 1$, there exists a constant $C > 0$ such that

$$\sup_K |s|^2 e^{-2kQ} \leq C e^{Ck^{1-\delta}} \|s\|^2_{L^2(\mu,kQ)}$$

for every $s \in \mathcal{P}_k$.

• (Dinh-Nguyên, 2018) If K is $(\mathcal{C}^{0,\alpha}, \mathcal{C}^{0,\alpha'})$-regular, and Q Hölder, and there exist constants $\tau > 0$, $r_0 > 0$ satisfying

$$\mu(\mathbb{B}(z,r) \cap K) \geq r^\tau$$

for every $z \in K$, $r \leq r_0$, then μ is an 1-Bernstein-Markov measure with respect to (K, Q).

- Let $d_k := \dim \mathcal{P}_k \approx k^n$.

- (Berman-Boucksom-Witt Nyström, 2010) One has

$$d_k^{-1} B_k \mu \to \mu_{K,Q}\,, \quad k \to \infty, \tag{2}$$

provided that μ is a Bernstein-Markov measure associated to (K, Q).

- For $n = 1$ and μ Berstein-Markov whose support $\mathrm{supp}\mu$ is a smooth Jordan arc in $\mathbb{C}$: Pointwise limit almost everywhere (Danka-Totik 2015, Mate-Nevai-Totik, Totik, etc.):

$$kB_k^{-1}(x) \to f(x),$$

almost everywhere if $\mu = f\mu_{K,Q}$ and $f > 0$. Moreover if $f(x) = |x - x_0|^\alpha w(x)$ with continuous w and $w(x_0) > 0$, then

$$k^{1+\alpha} B_k^{-1}(x_0) \to \text{an explicit number.}$$

Theorem (Marinescu-V, 2023)

Let K be a compact generic Cauchy-Riemann nondegenerate $\mathcal{C}^5$ piecewise-smooth submanifold in $\mathbb{C}^n$ of dimension n_K. Let Q be a Hölder continuous function of Hölder exponent $\alpha \in (0,1)$ on K, and let Leb_K be a smooth volume form on K, and $\mu = \rho Leb_K$, where $\rho \geq 0$ and $\rho^{-\lambda} \in L^1(Leb_K)$ for some constant $\lambda > 0$. Then we have

$$\sup_K B_k \leq C k^{2n_K(\lambda+1)/(\alpha\lambda)},$$

for some constant $C > 0$ independent of k.

- Consider $\lambda \to \infty$, $\alpha \to 1$, $n_k = 2n$: $\sup_K B_k \lesssim_\delta k^{2n+\delta}$ close to be optimal. If $n_k = n$, then $\sup_K B_k \lesssim_\delta k^{n+\delta}$ close to be optimal (compare to Totik et al).

- $\alpha \to 1$ means Q almost Lipschitz. In the standard smooth compact setting Q must be at least $\mathcal{C}^2$: in this case, $B_k \approx k^n$.

Upper bounds for Bergman's function

Theorem (Marinescu-V, 2023)

Let μ be a smooth volume forms of a maximally totally real submanifold K and $Q \in \mathcal{C}^{1,\delta}$ for some $\delta > 0$ (e.g., $Q \equiv 0$). Then its Bergman kernel function satisfies

$$\sup_K B_n \leq Ck^n.$$

• known if K smooth real algebraic (Berman-Ortega-Cerda, 2018) (it was proved there also that $B_n \gtrsim k^n$ if additionally $Q \equiv 0$).

• Key ingredients: Hölder regularity of $V_{K,Q}$, and suitable families of analytic discs partly attached to K.

• Let $s_1, \dots, s_{d_k}$ be an orthonormal basis of $\mathcal{P}_k(K)$ with respect to the $L^2(\mu, kQ)$-scalar product.
• Let

$$p_k := \sum_{j=1}^{d_k} \alpha_j s_j, \tag{3}$$

where α_j's are complex i.i.d. random variables.

• The most classical example may be the Kac polynomial where $n = 1$, $s_j = z^j$, $K = \mathbb{S}^1$ and μ the arc-length measure on $\mathbb{S}^1$.

• (Shiffman-Zelditch, 2003) The case $n = 1$, K a bounded domain with analytic boundary or K a closed analytic curve: Asymptotic expectation of zeros. Links to random matrix theory. Many prior and follow-ups results.

• (Bloom-Levenberg, 2015) If (K, Q, μ) is Bernstein-Markov, then almost surely

$$k^{-1}[p_k = 0] \to dd^c \log |V^*_{K,Q}|$$

as $k \to \infty$. Many subsequent results by others: Bayraktar, Coman, Marinescu, etc.

• Assume the distribution of α_j is $f\, Leb_{\mathbb{C}}$ and

(H1) $|f(z)| \leq |z|^{-3}$ for $|z|$ sufficiently large.

(H2) K be a non-degenerate piecewise-smooth generic Cauchy-Riemann submanifold of $\mathbb{C}^n$, and Q a Hölder continuous function on K, μ a smooth measure on K.

Zeros of random polynomials

Corollary (Marinescu-V, 2023)

Assume (H1)+ (H2). We have

$$E_k\left(k^{-1}[p_k=0]\right) = dd^c V_{K,Q} + O\left(\frac{\log k}{k}\right). \tag{4}$$

A large deviation type estimate is also obtained.

- If $n=1$, the equality (4) was proved by Shiffman-Zelditch with error term $O(k^{-1})$.
- (Marinescu-V, 2023) Let

$$\tilde{B}_k(x) := \sum_{j=1}^{d_k} |s_j(x)|^2 = \sup_{s\in\mathcal{P}_k} |s(x)|^2/\|s\|^2_{L^2(\mu,kQ)}$$

for $x\in\mathbb{C}^n$. If (H2) holds, then there exists constant $C>0$ such that

$$\left\|\frac{1}{2k}\log\tilde{B}_k - V_{K,Q}\right\| \leq C\frac{\log k}{k}.$$