Complex Hessian equations and their applications

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Elementary symmetric polynomials

- Gårding '59 studied hyperbolic polynomials. A polynomial P of degree k in $\mathbb{R}^n$ is hyperbolic with respect to a if for any b $t \to P(b+at)$ has exactly k real roots. His results applied to elementary symmetric polynomial of degree m in λ:

$$S_m(\lambda) = \sum_{0<j_1<\ldots<j_m\leq n} \lambda_{j_1}\lambda_{j_2}, \ldots\lambda_{j_m}.$$

imply that the "positive cone" Γ_m $(m \leq n)$

$$\Gamma_m = \{\lambda \in \mathbb{R}^n |\ S_1(\lambda) > 0,\ \cdots,\ S_m(\lambda) > 0\}$$

is convex and the function $S_m^{\frac{1}{m}}$ is concave when restriced to Γ_m.

Elementary symmetric polynomials

- In the results that follow many other inequalities relating $S_m(\lambda)$ and $S_{k;i}(\lambda) := S_k(\lambda)_{\lambda_i=0} = \frac{\partial S_{k+1}}{\partial \lambda_i}(\lambda)$ are applied. In particular:

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- Newton - Maclaurin's inequality: If $\lambda \in \Gamma_m$ then

$$\left(\frac{S_j}{\binom{n}{j}}\right)^{\frac{1}{j}} \geq \left(\frac{S_i}{\binom{n}{i}}\right)^{\frac{1}{i}}$$

for $1 \leq j \leq i \leq m$;

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  for $1 \leq j \leq i \leq m$;
- For any $\lambda,\ \mu \in \Gamma_m$

$$\sum_{i=1}^{n} \mu_i S_{m-1;i}(\lambda) \geq m S_m(\mu)^{\frac{1}{m}} S_m(\lambda)^{\frac{m-1}{m}}.$$

- Ivochkina's characterisation of the cone Γ_m tells that if $\lambda \in \Gamma_m$, then

$$S_{k;i_1,\ldots,i_t}(\lambda) > 0$$

for all $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, n\}$, $k + t \leq m$. In particular, if $\lambda \in \Gamma_m$, then at least m of the numbers $\lambda_1, \ldots, \lambda_n$ are positive.

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- Lin-Trudinger inequality: there exists a constant θ, depending on n and m, such that for any $\lambda \in \Gamma_m$ and $i \geq m$

$$\frac{S_{m;i}(\lambda)}{S_m(\lambda)} \geq \theta.$$

(It does not hold for $i < m$.)

- Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be the eigenvalues of $n \times n$ Hermitian matrix A. $S_m(A)$ denotes the elementary symmetric polynomial of degree m in λ:

$$S_m(A) = \sum_{0<j_1<...<j_m \leq n} \lambda_{j_1} \lambda_{j_2}, ... \lambda_{j_m}.$$

If f is a given function on $\Omega \subset \mathbb{C}^n$, and A is the complex hessian of the unknown u of class C^2 then

$$S_m\big((u_{z_j \bar{z}_k}(z))\big) = f(z)$$

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Complex m-Hessian equations in $\mathbb{C}^n$

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- In terms of differential forms:

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n, \quad \beta = dd^c|z|^2.$$

Complex m-Hessian equations in $\mathbb{C}^n$

- The equation is elliptic on the set of m-subharmonic functions (m-sh). A $\mathcal{C}^2(\Omega)$ function u is called m-sh if for any $z \in \Omega$ the Hessian matrix $\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}(z)$ has eigenvalues in the closure of the cone Γ_m. In other words

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- The corresponding equation for real Hessian studied since eighties by Caffarelli-Nirenberg-Spruck, Ivochkina, Trudinger-Wang, many others. For complex dimension 1 we recover the Poisson equation, for top dimension n it is the complex Monge-Ampère equation. The latter is special since this is the only case when we deal only with nonnegative eigenvalues. Nevertheless one can try to adapt the methods working for real Hessian or complex Monge-Ampère equations. In particular pluripotential theory can be extended to m-sh functions (Błocki).

- Following the method of Caffarelli-Nirenberg-Spruck for real Hessian equation Vinacua '86, S. Y. Li '06 solved the Dirichlet problem:

Theorem

Let Ω be a smoothly bounded relatively compact domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is $(m-1)$-pseudoconvex (that means that Levi form at any point $p \in \partial\Omega$ has its eigenvalues in the cone Γ_{m-1}). Let φ be a smooth function on $\partial\Omega$ and f a strictly positive and smooth function in Ω. Then the Dirichlet problem

$$\begin{cases} u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega}); \\ (dd^c u)^m \wedge \beta^{n-m} = f \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a smooth solution u.

Complex m-Hessian equations in $\mathbb{C}^n$

- The pluripotential approach was introduced by Błocki who defined m-sh function as a subharmonic function satisfying

$$dd^c u \wedge \alpha_1 \wedge \ldots \wedge \alpha_{m-1} \wedge \beta^{n-m} \geq 0$$

for any collection of m-positive $(1,1)$ forms α_j (α is m-positive if $\alpha^j \wedge \beta^{n-j} \geq 0$ for $j = 1, \ldots, m$). He established some basic properties of those functions and studied their integrability.

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- Conjecture of Błocki: any m-sh function belongs to $L^q_{loc}(\Omega)$ for any $q < \frac{mn}{n-m}$. The exponent cannot be beter since

$$G(z) = -|z|^{2-2n/m}$$

  is m-sh and belongs to L^q_{loc} if and only if $q < \frac{mn}{n-m}$. Conjecture confirmed by Dinew-K. under extra hypothesis that the function is bounded near $\partial\Omega$. In general still open.

Complex m-Hessian equations in $\mathbb{C}^n$

- Having this integrability result Dinew-K. obtained new volume -capacity estimate and further the existence of weak solutions of the Dirichlet problem

Theorem

Let Ω be smoothly bounded (m-1)-pseudoconvex domain. Then for $q > n/m$, $f \in L^q(\Omega)$ and continuous φ on $\partial\Omega$ there exists $u \in SH_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$ satisfying

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- For $\varphi \in C^{1,1}$ the above solutions are Hölder continuous by C. N. Nguyen (with extra assumption) and Charabati.
- Those proofs follow the ones for the M-A equation with some variations.

- Similarly, Ch. Lu adapted Cegrell's "finite energy" method showing that for a positive measure of finite total mass and vanishing on $-\infty$ sets of m-sh functions on the RHS of the m-Hessian equation one can find possibly unbounded solution.

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- Summarizing: for smooth data real Hessian equations methods work, for non-smooth degenerate data pluripotential tools can be adapted. However the space of m-sh functions is less understood than psh functions. For instance m-sh function is subharmonic on any $n - m + 1$ affine subspace, but it is not a charcterization.

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- When we pass to compact manifolds life becomes harder.

Hessian equations on Kähler manifolds

- Hessian equations on compact Kähler manifolds

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n, \quad \int_X f\omega^n = \int_X \omega^n$$

(ω a Kähler form on compact manifold X, $f \geq 0$ the given function, $1 < m < n$) The solutions should be (m, ω)-sh and satisfy

$$(\omega + dd^c u)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, ..., m.$$

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- The solution of this equation generalizes Calabi-Yau theorem. In fact the continuity method from Yau's proof can be adapted (not directly) in case of non-negative bisectional curvature (Hou, Jbilou, Kokarev). The missing part in the general case are C^2 estimates.

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- Hou-Ma-Wu refined a method of Chou and Wang (real Hessian equations on Riemanian manifolds) reducing the problem to C^1 estimate. They proved the following.

Theorem

If $u \in C^4$ solves the above equation then for an independent C

$$\sup_X ||dd^c u||_\omega \leq C(\sup_X ||\nabla u||^2 + 1).$$

- Hou-Ma-Wu also suggested a use of blowing-up analysis, which transforms the given equation in a coordinate patch into homogeneous Hessian equation in $\mathbb{C}^n$.

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- The last step was made by Dinew and K. (with "pluripotential proof")

Theorem

Any bounded, maximal m-sh function with bounded gradient in $\mathbb{C}^n$ is constant. Therefore the Hessian equation on a compact Kähler manifold is solvable for smooth, positive f.

- Also using similar argument as in M-A case Dinew, K. proved that there exist weak continuous solutions of the equation for $f \in L^p(\omega^n),\ p > n/m$, and the bound on p is sharp.
Those solutions are Hölder continuous in case of nonnegative holomorphic bisectional curvature, by Cheng and Xu.

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- Variants of Hou-Ma-Wu estimate and Dinew-K. Liouville type argument were subsequently used in many proofs of more general equations mentioned below.

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- Variants of Hou-Ma-Wu estimate and Dinew-K. Liouville type argument were subsequently used in many proofs of more general equations mentioned below.
- Recently Guo-Phong-Tong applied different method (purely PDE) to prove L^∞ estimates for this theorem in a more general setting.

- Now ω is not closed on a compact complex mfd X. We wish to find a constant c and (m,ω)-sh such that

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = cf\omega^n.$$

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- $d\omega \neq 0$ makes a lot of difference. First c is not known a priori. The comparison principle is not true and then the uniqueness of weak solutions becomes nontrivial.
- Existence in the smooth non-degenerate case is due to Tosatti - Weinkove for the M-A equation. In general to D. Zhang and G. Székelyhidi independently. The Kähler case methods are adapted, but considerably refined by new inequalities for symmetric polynomials (first author), a new theory of $\mathcal{C}$-subsolutions (B. Guan, second author).

- In a joint paper with C. N. Nguyen we also developed the "pluripotential" approach to (m, ω)-sh (but only continuous) functions. In particular we obtained continuous solutions of the equation for $f \in L^p(\omega^n),\ p > n/m$.
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  This result was recently extended by Guedj and Lu to background forms which are semipositive and big, using a different proof.
- Except for problms mentioned above, note that (m, ω)-sh functions do not form a cone which makes constructions and regularizations more difficult. In particular smoothing by convolution even in charts does not preserve (m, ω)-subharmonicity. Just recently in an ongoing project we were able to define the m-Hessian measure for a bounded (m, ω)-sh function. Those results required some new inequalities for symmetric polynomials.

Hessian type equations

- Equation discussed above can be generalized in many ways:
  1. more general differential operator
  2. more general background form
  3. more general right hand side

  and combinations of those, including coupled equations.

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- with eigenvalues of the relative endomorphism between ω (or other fixed form) and $\omega + dd^c u$

- Some examples:
  J- equation: $\omega^n = \omega^{n-1} \wedge \chi$, where χ is fixed Kähler form, and ω the searched Kähler form. It can be expressed as

$$\frac{S_{n-1}}{S_n}(\lambda) = n.$$

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- Hessian quotient equations:

$$(\omega + dd^c u)^k \wedge \chi^{n-k} = g(z)(\omega + dd^c u)^m \wedge \chi^{n-m}, \quad k < m < n.$$

  solved by G. Székelyhidi for constant g and W. Sun for smooth positive g.

Hessian type equations

- p-Monge-Ampère equation of Harvey and Lawson:

$$f(\lambda) = \prod_I \lambda_I^{\frac{n!}{(n-p)!p!}},$$

where I runs over all multi-indices $1 \le i_1 < ... < i_p \le n$.
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- The method of those authors works also for more general background forms. Suppose (X, ω) is compact, Kähler, and $[\chi]$ is a nef cohomology class on X. Then for each $t \in (0, 1]$ the class of $\chi + t\omega$ is Kähler. Now, the point is that a priori estimates for family of equations indexed by t are independent of the parameter if, for instance, RHS is in $f \in L^p(\omega^n)$, $p > n/m$.

Hessian type equations

- Fu-Yau '08 proved the existence of some special solutions to the Hull-Strominger system (supersymmetry in string theory) reducing the system to M-A type equation in dimension 2. Those solutions produce a toric fibration over a $K3$ surface. In higher dimensions such a reduction leads to hessian type equation with a gradient term:

$$(\chi(z,u) + dd^c u)^k \wedge \omega^{n-k} = \psi(z,u,\nabla u)\omega^n,$$

where $\chi(z,u)$ is a (1,1) form. Phong-Picard-Zhang '14-'19 solved those equations under mild assumptions. Another proof of the last item given by J. Chu, L. Huang and X. Zhu.

- A Gauduchon metric satisfies: $dd^c\omega^{n-1}=0$. Such metrics exist on any complex manifold. The Gauduchon conjecture is the analogue of the Calabi conjecture and (roughly) says that one can find a Gauduchon metric whose Chern-Ricci form is equal to a given form. Székelyhidi-Tosatti-Weinkove '18 confirmed this conjecture solving a twisted hessian equation. Suppose ω, ω_0 are given Gauduchon metrics, F a smooth function on a compact complex manifold, then there exists a smooth solution u and a constant b such that a Hermitian form defined by

$$\alpha^{n-1}=\omega_0^{n-1}+dd^c u\wedge\omega^{n-2}+\Re(i\partial u\wedge\bar\partial(\omega^{n-2})),$$

satifies

$$\alpha^n=e^{F+b}\omega^n$$

.

- For a smooth function $u : \mathbb{R}^n \to \mathbb{R}$ the graph $(x, \nabla u(x))$ is called special Lagrangian if it is area minimizing among all submanifolds with the same boundary. Harvey and Lawson proved that the gradient graph is area minimizing iff the eigenvalues of the hessian of u satisfy the equation

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$$\sum \arctan \lambda_j = \theta.$$
- In relation to the mirror symmetry for Calabi-Yau manifolds Collins-Jacob-Yau '18 solved a corresponding equation assuming the existence of subsolutions. If ω is a Kähler form, we look for a real smooth, closed $(1,1)$ form χ such that
$$\Im((\omega + i\chi)^n) = const.\Re((\omega + i\chi)^n).$$
The solution yields special Lagrangian surfaces in Calabi-Yau manifolds.

- THANK YOU and ...

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- A ROUND OF APPLAUSE FOR THE ORGANIZERS!