Obstruction Flat Rigidity of the CR 3-Sphere

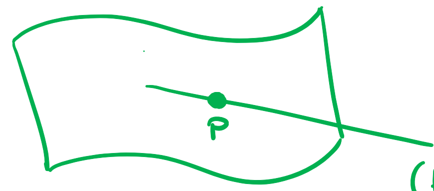
Sean Curry

Oklahoma State University

* Joint with P. Ebenfelt

Complex Analysis, Geometry
and Dynamics
Portorož 2023

CR 3-Manifolds



$M^3 \subseteq \mathbb{C}^2$ real hypersurface

$\rightsquigarrow$ CR structure (M, H, J)

- (generically) H contact distribution
  $\hookrightarrow$ $H = \ker \theta$, θ 1-form
  $\theta \wedge d\theta \neq 0$
- $J : H \to H$, $J^2 = -\mathrm{Id}$.

Notation: $(M, H, J) \leftrightarrow (M, T^{1,0}) \leftrightarrow (M, \bar{\partial}_b)$

$$\mathbb{C}H = \underset{i}{T^{1,0}} \oplus \underset{-i}{T^{0,1}} \quad \text{eigenspaces of } J$$

Often $T^{1,0} = \operatorname{span}\{Z_1\}$ "$\bar{\partial}_b = Z_{\bar{1}} = \overline{Z_1}$"

! $[T^{1,0}, T^{1,0}] \subseteq T^{1,0}$ trivially.

CR Structure of Unit $S^3 \subseteq \mathbb{C}^2$

$$u = 1 - |z|^2 - |w|^2, \qquad \Theta = i\partial u|_{TS^3}$$

On S^3

- $\Theta = i(z\,d\bar{z} + w\,d\bar{w})|_{TS^3}$ (real)
- $d\Theta = i(dz \wedge d\bar{z} + dw \wedge d\bar{w})$
- $Z_1 = \bar{w}\partial_z - \bar{z}\partial_w$
- $Z_{\bar{1}} = w\partial_{\bar{z}} - z\partial_{\bar{w}}$
- $T = i(z\partial_z + w\partial_w) - i(\bar{z}\partial_{\bar{z}} + \bar{w}\partial_{\bar{w}})$
  (Reeb v.f. of Θ)

$[Z_1, Z_{\bar{1}}] = -iT$

$[T, Z_1] = -2iZ_1$

$[T, Z_{\bar{1}}] = 2iZ_{\bar{1}}$

Abstract Deformations

- Gray's theorem: might as well fix H. $\rightarrow$ only vary J.

$|\varphi|^2 < 1$ (pointwise)

$$\rightsquigarrow \quad \hat{Z}_1 = Z_1 + \varphi_1{}^{\bar{1}} Z_{\bar{1}} \qquad \varphi_1{}^{\bar{1}} \in C^\infty(S^3, \mathbb{C})$$

CR Obstruction Flatness ($\mathcal{O} \equiv 0$)

Chern-Moser vs Bergman/Szegö

Fefferman ('76, '79) proposed the development of CR invariant theory through the study of the formal asymptotics of the following (biholomorphically invariant) Dirichlet problem:

$$\begin{cases} J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar z^k} \\ u_{z^j} & u_{z^j \bar z^k} \end{pmatrix} = 1 & \text{in } \Omega \subseteq \mathbb{C}^n \\ (u > 0 \text{ in } \Omega) \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

bdd., str. ψ-cvx.

$v = -\log u$ is a Kähler potential for ! complete K-E metric

$(\det(v_{z_j \bar z_k}) = e^{(n+1)v})$

- Cheng-Yau: $\exists!$ solution $u \in C^\infty(\Omega) \cap C^{n+1}(\bar\Omega)$
- Lee-Melrose: $u \sim \rho + \underline{\mathcal{O}} \rho^{n+2} \log \rho + \ldots$

  a local CR invariant

  ρ a Fefferman def. function $J(\rho) = 1 + O(\rho^{n+1})$
- Graham: $\mathcal{O} \equiv 0 \Rightarrow u \in C^\infty(\bar\Omega)$

  $\leftarrow$ "CR obstruction density" (Many roles!)

*Physics: Bach flat (conf. gravity) & AdS/CFT

Solutions to $\mathcal{O} \equiv 0$?

Cauchy-Kowalewski

Set $\mathcal{O} \equiv 0$ up as a Cauchy problem with appropriate initial data.

* Local:

  - Graham '87: $\exists$ infinitely many (inequivalent) local real analytic hypersurfaces in $\mathbb{C}^n$ $(\forall n \geq 2)$ with $\mathcal{O} \equiv 0$.

* Global (compact):

  - $u = 1 - \|z\|^2$ on the ball, so the CR spheres are obstruction flat (as are locally spherical structures).
  - That is all we know...

* Rigidity Results:

  (non-perturbative) • C.-Ebenfelt '18: $S^3 \subseteq \mathbb{C}^2$ has no obstruction flat deformations inside $\mathbb{C}^2$.

  what about outside?

  (perturbative) • Hirachi-et. al. (in preparation): $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ ...

Hope!

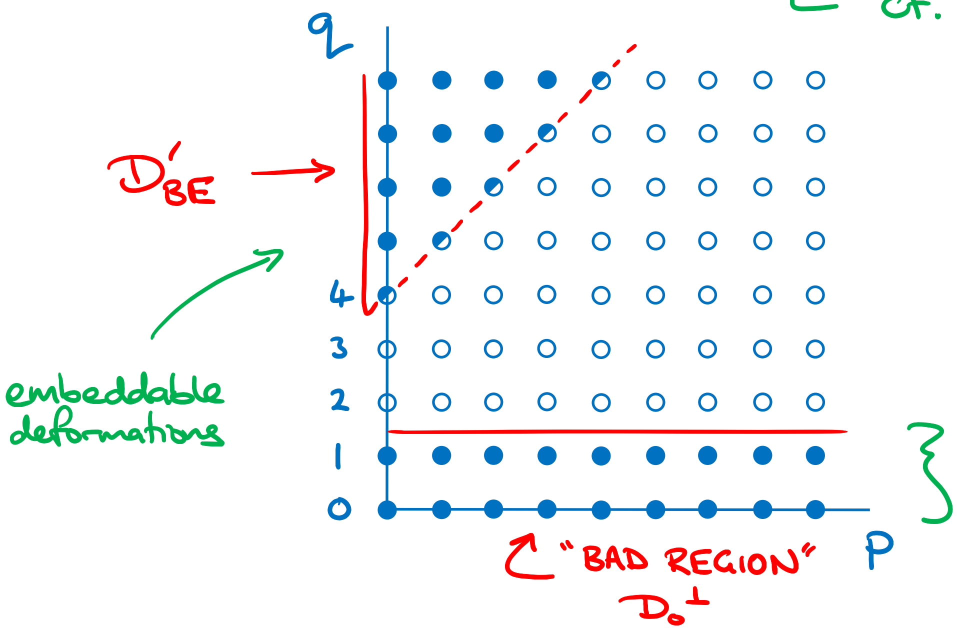
spherical harmonics:

$H_{p,q} \subseteq P_{p,q}$

(e.g., $(z-w)^p(\bar{z}+\bar{w})^q \in H_{p,q}$)

Slice Theorem (C.-Ebenfelt '20):

← cf. Burns-Epstein, Bland, Cheng-Lee



Any sufficiently small abstract deformation of the unit $S^3 \subseteq \mathbb{C}^2$ is equivalent to one in this slice (unique mod $\mathrm{Aut}_{CR}(S^3)$).

← $PSU(2,1)$

kernel of the linearized obstruction operator $D\Theta$!

↑

infinite dimensional & complementary to the embeddable deformations!

⇝ Deformation Theory! Obstructed?

Dashed!

Theorem (C.-Ebenfelt '21):

There is an open neighborhood $\mathcal{U}$ of the origin in the slice $D'_{BE} \oplus D_0^{\perp}$ such that the CR structure corresponding to $\varphi_1^{\top} \in \mathcal{U}$ is obstruction flat if and only if $\varphi_1^{\top} = 0$.

* Topology: C^3 (or better H^3_{FS})

$\longrightarrow$ There are no solutions of $\mathcal{O} \equiv 0$ near (S^3, H_{std}, J_{std}), embeddable or nonembeddable.

Indication of Proof (by contradiction):

Suppose $\varphi^{(k)} \xrightarrow[H^3_{FS}]{} 0$ solves $\Theta(\varphi^{(k)}) = 0 \ \forall k \in \mathbb{N}$.

① set $\varphi^{(k)} = \varepsilon_k \hat{\varphi}^{(k)}$ with $\|\hat{\varphi}^{(k)}\|_{H^3_{FS}} = 1$ $(\varepsilon_k \to 0)$

6th order in φ

② $\Theta = \nabla'\nabla' Q_{11} - iA'' Q_{11}$. Show (WORK) $\underbrace{\|\hat{\varphi}^{(k)}_{D'_{BE}}\|_{H^3_{FS}} \to 0}_{\|\hat{\varphi}^{(k)}_{D_0^\perp}\|_{H^3_{FS}} \to 1}$

$\Theta(\varphi^{(k)}) = 0 \rightsquigarrow \underline{\underline{D\Theta}}(\varphi^{(k)}_{D'_{BE}}) = \mathcal{F}(\varphi^{(k)})$

injective on D'_{BE} and inverse gains 6 derivatives in Folland-Stein spaces.

nonlinear terms (involve 6 derivatives but we only control 3!)

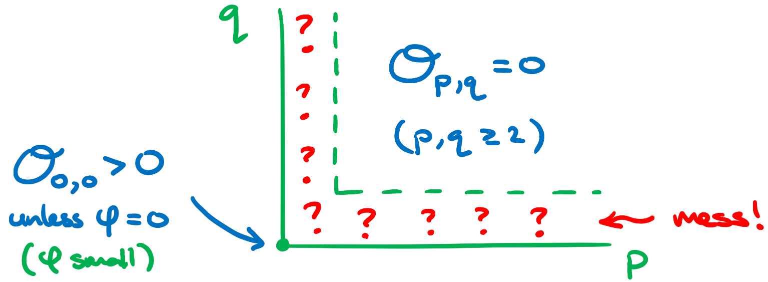
③ $\int_{S^3} \Theta(\varphi^{(k)}) \sim \varepsilon_k^2 \|\hat{\varphi}^{(k)}_{D_0^\perp}\|^2_{H^3_{FS}} + O(\varepsilon_k^{5/2})$

$\hookrightarrow$ So $0 = \lim_{k\to\infty} \frac{1}{\varepsilon_k^2} \int \Theta(\varphi^{(k)}) = 1$ ↯ $\square$

* Formally: we can solve to first order, but not to second order as the $iA'' Q_{11}$ term "wants to be positive" (on average).

Concluding Remarks:

- One can solve $\Theta \equiv 0 \mod \operatorname{coker}(D\Theta)$, i.e. $\Theta_{p,q} = 0$ for $p,q \geq 2$. Every $\dot{\varphi} \in D_0^{\perp}$ integrates to a family $\varphi(t)$ of solutions of this equation (with $\varphi(0) = 0$ & $\frac{d}{dt}\big|_{t=0} \varphi(t) = \dot{\varphi}$).



- Our result proves linearization instability for the conformal Einstein static universe $S^3 \times \mathbb{R}$ in the space of Bach flat metrics. ("Expected." Cf. Moncreif, Fischer-Marsden, and others in the Einstein case.)

Thank You!