On non-autonomous attracting basins.

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Let $\{F_n\} \in \mathrm{Aut}(\mathbb{C}^m)$ be a sequence which is uniformly attracting at the origin, i.e., there exists $r > 0$ such that for $z \in B(0;r)$

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where $0 < A < B < 1$. Then

$$\Omega^0_{\{F_n\}} = \{z \in \mathbb{C}^m : F(n)(z) = F_n \circ \cdots \circ F_1(z) \to 0 \text{ as } n \to \infty\} \simeq \mathbb{C}^m.$$

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such that the below diagram commutes upto jets of order d_0. Then $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{H_n\}}$, where $\Omega^0_{\{H_n\}}$ denotes the **abstract** basin of attraction of the sequence $\{H_n\}$ at the origin.

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- For every $n \geq 1$, $\pi_1 \circ F_n(x,y) = a_n x + e_n^1(x) + f_n^1(x,y) + g_n^1(y)$

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- The triangularisation technique fails, unless $g_n^1(y) \equiv 0$ or $B^2 < A$ or $|a_n|^2 < |b_n|$.

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where e_n^i, g_n^i are holomorphic in ***one*** variable and f_n^i are holomorphic in ***two*** variables for $i = 1,2$.

- The triangularisation technique fails, unless $g_n^1(y) \equiv 0$ or $B^2 < A$ or $|a_n|^2 < |b_n|$.

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Proof of Step 1: $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{H_n\}}$.

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- Step 1: Any uniformly attracting sequence of automorphisms is non-autonomously conjugated with a sequence of Hénon maps of bounded degree.

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Let $\{F_n\} \in \mathrm{Aut}(\mathbb{C}^2)$ be a sequence which is uniformly attracting at the origin with $B^{d_0} < A < B$ for some $d_0 \geq 2$. Then the sequence $\{F_n\}$ is **non-autonomously conjugated** to a sequence of uniformly attracting Hénon maps $\{H_n\}$ each of degree $d_0 + 1$, i.e., $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{H_n\}}$.

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- **Definition:** A finite composition of maps of the form

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where $p(y)$ is a polynomial of degree at least 2 and $a \in \mathbb{C}^*$.

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- **Result**

  - The sequence $\{G_n^{\pm}\}$ converges uniformly to $G_{\{H_n\}}^{\pm}$ over compact subsets of $\mathbb{C}^2$.
  - $dd^c G_n^{\pm} \to dd^c\left(G_{\{H_n\}}^{\pm}\right)$.

$\Omega^0_{\{H_n\}}$ is biholomorphic to $\mathbb{C}^2$.

- Let $C \subset \Omega^0_{\{H_n\}}$ is a compact subset then there exists $n_C \geq 1$ such that
$$C \subset \Omega_n \text{ for every } n \geq n_C.$$
- Let $\{C_l\}, l \geq 1$ be an exhaustion by compacts of $\Omega^0_{\{H_n\}}$, then there exist an increasing sequence of integers $\{n_l\}$ such that
$$C_l \subset \Omega_n \text{ for every } n \geq n_l \text{ and } l \geq 1.$$
- We consider **appropriately normalised** biholomorphisms $\phi_l : \Omega_{n_l} \to \mathbb{C}^2$ such that
$$\lim_{l\to\infty} \phi_l(z) \text{ exists for every } z \in \Omega^0_{\{H_n\}}.$$
- Also $\phi = \lim_{l\to\infty} \phi_l$ on $\Omega^0_{\{H_n\}}$ is an injective map.
- Finally, by the fact— $G_n^+ \to G^+_{\{H_n\}}$ (uniformly on compacts) we conclude $\phi(\Omega^0_{\{H_n\}}) = \mathbb{C}^2$.

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where $a \neq 0$ and $p : \mathbb{C}^{m-1} \to \mathbb{C}$ is a polynomial map of degree at least 2.

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- (Technical) Step 1: A uniformly attracting sequence $\{F_n\}$ is non-autonomously conjugated to a sequence $\{G_n\}$ where every $\{G_n\}$ is obtained as $m-$composition of weak shift-like maps, i.e.,

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and $\{S_n\}$ is a sequence of weak shift-like maps of degree d_0 as jets of order $d_0 \geq 2$.

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where $a \neq 0$ and $p : \mathbb{C}^{m-1} \to \mathbb{C}$ is a polynomial map of degree at least 2.

- (Technical) Step 1: A uniformly attracting sequence $\{F_n\}$ is non-autonomously conjugated to a sequence $\{G_n\}$ where every $\{G_n\}$ is obtained as $m-$composition of weak shift-like maps, i.e.,

$$G_n = S_{mn} \circ \cdots \circ S_{m(n-1)+1},$$

and $\{S_n\}$ is a sequence of weak shift-like maps of degree d_0 as jets of order $d_0 \geq 2$.

$$\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}} = \Omega^0_{\{S_n\}}.$$

Idea of Proof of (Technical) Step 1 in $\mathbb{C}^3$: $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}}$.

Idea of Proof of (Technical) Step 1 in $\mathbb{C}^3$: $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}}$.

- We will assume $DF_n(0) = \begin{pmatrix} a_n & 0 & 0 \\ d_n & b_n & 0 \\ e_n & f_n & c_n \end{pmatrix}$, with $0 < A \leq |a_n| \leq |b_n| \leq |c_n| \leq B < 1$ with $B^{d_0} < A$.
- For every $n \geq 1$,

$$\pi_1 \circ F_n(x,y) = a_n x + a_n^x(x,y,z) + a_n^2(y) + a_n^3(z) + a_n^{23}(y,z),$$

$$\pi_1 \circ F_n(x,y) = b_n y + d_n x + b_n^y(x,y,z) + b_n^1(x) + b_n^3(z) + b_n^{13}(x,z),$$

$$\pi_1 \circ F_n(x,y) = c_n z + e_n x + f_n y + c_n^z(x,y,z) + c_n^1(x) + c_n^2(y) + c_n^{12}(x,y),$$

where $a_n^{[\cdot]}$, $b_n^{[\cdot]}$ and $c_n^{[\cdot]}$ are appropriately defined holomorphic functions.

- Observe that linear part obtained as composition of weak-shifts is straightforward.

Idea of Proof of (Technical) Step 1 in $\mathbb{C}^3$: $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}}$.

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- Observe that linear part obtained as composition of weak-shifts is straightforward.
- The **triangularisation** technique fails, unless all the terms in red are identically equal to **zero.**
- Next the goal is to obtain degree **two** part of the red terms , above.
- We calculate in the same way the degree **three** terms of the red part, considering the errors induced by the degree **two** part. The process continues inductively upto the $d_0 + 1$-th stage.
- The sequences $\{G_n\}$ and $\{F_n\}$ are non-autonomously conjugated, upto jets of order $d_0 + 1$, i.e., $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}}$.

Outline of the proof of the Bedford's Conjecture in $\mathbb{C}^m, m > 2$.

- **Step 1:** $\Omega^0_{\{F_n\}} \simeq \Omega^0_{\{G_n\}} = \Omega^0_{\{S_n\}}$.

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and $\{\mathbf{S_n}\}$ is a $(d_0 + 2)$-perturbation of the **above sequence,** $\{S_n\}$ as jets of order $d_0 \geq 2$.

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$d-$Perturbed weak shift-like maps in $\mathbb{C}^m, m \geq 3$.

$$S(z_1, \ldots, z_m) = \left(z_2, \ldots, z_m + z_2^{d-1}, az_1 + p(z_2, \ldots, z_m) + (z_2^d + \cdots + z_m^d)\right)$$

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  Hence one can mimic the ideas as in the case of Hénon to actually prove $\Omega_{\{\mathbf{S_n}\}} \simeq \mathbb{C}^m$.

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