A flower theorem in dimension two

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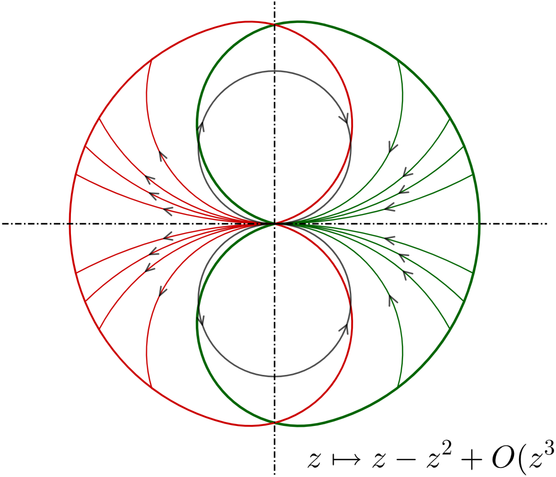
Objective

We study the local dynamics of biholomorphisms in $\mathbb{C}^2$ tangent to the identity, i.e. $F(z) = z + \dots$

In dimension one, the dynamics is well understood:

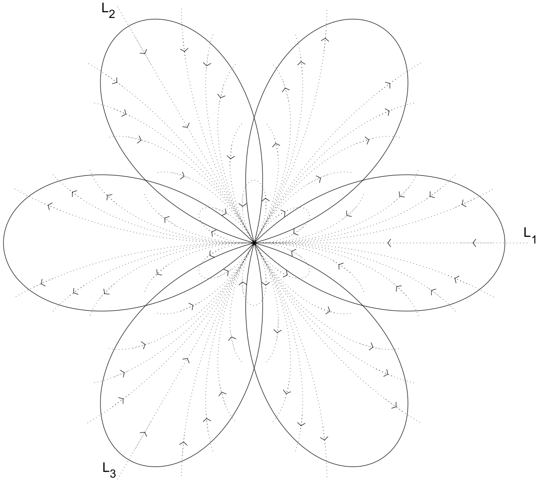
Theorem (Leau-Fatou flower theorem)

If $F(z) = z + az^{k+1} + ...$, $a \neq 0$, there are $2k$ sectorial domains (called petals), covering a punctured neighborhood of the origin, where the orbits are alternatively attracted or repelled by the origin. Moreover, for each petal Ω there exists a holomorphic injective map $\varphi : \Omega \to \mathbb{C}$ such that $\varphi \circ F \circ \varphi^{-1}(z) = z + 1$.



$$z \mapsto z - z^2 + O(z^3)$$

Dynamics in dimension one



Dynamics of $z \mapsto z - z^4 + z^5 (\ldots)$

Dimension two

If F is a tangent to the identity biholomorphism in $(\mathbb{C}^2, 0)$, we write

$$F(x, y) = (x + p_{k+1}(x, y) + \dots, y + q_{k+1}(x, y) + \dots),$$

where p_j, q_j are homogeneous polynomials of degree j and $(p_{k+1}, q_{k+1}) \not\equiv 0$.

A characteristic direction of F is a direction $[v] \in \mathbb{P}_{\mathbb{C}}$ such that

$$(p_{k+1}(v), q_{k+1}(v)) = \lambda v$$

for some $\lambda \in \mathbb{C}$. They are the only directions along which some stable dynamics of F can exist: if an orbit of F converges tangentially to a direction $[v]$, then $[v]$ is a characteristic direction.

Parabolic curves in dimension two

A parabolic curve is a connected domain $\Omega \subset \mathbb{C}^2$ of dimension 1 such that $0 \in \partial\Omega$, $F(\Omega) \subset \Omega$, $F^n(p) \to 0$.

Theorem (Écalle, Hakim)

If $[v]$ is a non-degenerate characteristic direction of F (i.e. $\lambda \neq 0$), then there are at least k parabolic curves of F where the orbits converge tangentially to $[v]$. (Holds in arbitrary dimension)

Theorem (Abate)

Either F has a curve of fixed points or there exists one characteristic direction $[v]$ supporting at least k parabolic curves.

Theorem (L.H., Rosas)

If $[v]$ is a characteristic direction of F, then there exists either a curve of fixed points tangent to $[v]$ or a parabolic curve where the orbits converge tangentially to $[v]$.

Parabolic domains in dimension two

A parabolic domain is a connected domain $\Omega \subset \mathbb{C}^2$ of dimension 2 such that $0 \in \partial\Omega$, $F(\Omega) \subset \Omega$, $F^n(p) \to 0$.

- Hakim: if $[v]$ is a non-degenerate characteristic direction and satisfies certain condition, then there are parabolic domains where the orbits converge tangentially to $[v]$. (Holds in arbitrary dimension)
- Vivas: classification of characteristic directions + conditions to ensure the existence of parabolic domains.

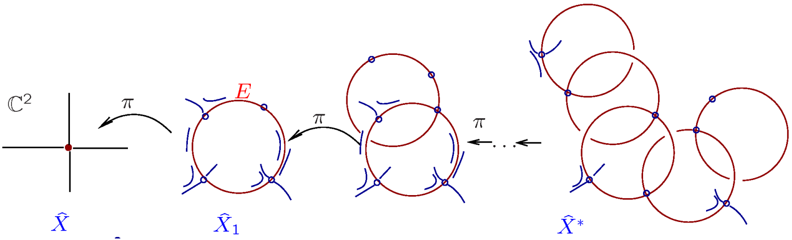
All the previous results are based on the resolution of F (Abate, Seidenberg)

Resolution of F

F is the time-1 flow of a formal vector field $\hat{X}$. After a sequence of blow-ups, the transform of X at each point can be written

$$\hat{X}^* = x^m y^n Y$$

where $m \geq 1$, $n \geq 0$ and Y either is non-singular or has a non-nilpotent linear part.



The transform of F under that sequence of blow-ups is the time-1 flow of $\hat{X}^*$.

Resolution of F

The transform of X at each point can be written

$$\hat{X}^* = x^m y^n Y$$

where $m \geq 1$, $n \geq 0$ and Y either is non-singular or has a non-nilpotent linear part. When Y is non-singular, there are no orbits converging to the fixed point. When Y is singular, if both eigenvalues are non-zero we say that 0 is a non-degenerate singularity; otherwise, it is a saddle-node.

We work at non-degenerate singularities. At these points, F can be written

$$F(x,y) = \left(x + x^{m+1}\left[a + \ldots\right], y + x^m\left[by + \ldots\right]\right)$$

if $n = 0$, or

$$F(x,y) = \left(x + x^{m+1}y^n\left[a + \ldots\right], y + x^m y^{n+1}\left[b + \ldots\right]\right)$$

if $n \geq 1$, with $m \geq 1$ and $ab \neq 0$ in both cases.

A toy model

We use the flow of the vector field

$$x^m y^n \left(ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} \right)$$

as a toy model for the dynamics. The orbits are

$$x(t) = x \, [1 - (am + bn) x^m y^n t]^{-a/(am+bn)}$$
$$y(t) = y \, [1 - (am + bn) x^m y^n t]^{-b/(am+bn)} \, ,$$

where $(x, y) = (x(0), y(0))$, so they converge to 0 if and only if

$$\operatorname{Re} \left(\frac{a}{am + bn} \right) > 0 \quad \text{and} \quad \operatorname{Re} \left(\frac{b}{am + bn} \right) > 0.$$

The flower theorem

Let 0 be a non-degenerate singularity of F. If $n = 0$, assume that F satisfies

$$\mathrm{Re}\,(b/a) > 0$$

and set $d = m$; if $n \geq 1$, assume that F satisfies

$$\mathrm{Re}\left(\frac{am+bn}{a}\right) > 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{am+bn}{b}\right) > 0$$

and set $d = (m, n)$.

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and set $d = (m, n)$. Then, in any neighborhood of the origin there exist d pairwise disjoint connected open sets $\Omega_0^+, \Omega_1^+, \ldots, \Omega_{d-1}^+$, and d pairwise disjoint connected open sets $\Omega_0^-, \Omega_1^-, \ldots, \Omega_{d-1}^-$, such that the following assertions hold:

- The sets Ω_k^+ are invariant for F and $F^j \to 0$ as $j \to +\infty$ on Ω_k^+, and the sets Ω_k^- are invariant for F^{-1} and $F^{-j} \to 0$ as $j \to +\infty$ on Ω_k^-.
- $\Omega_0^+, \ldots, \Omega_{d-1}^+, \Omega_0^-, \ldots, \Omega_{d-1}^-$, together with the fixed set $\{xy^n = 0\}$, cover a neighborhood of the origin.
- For each k, there exist injective holomorphic maps $\varphi_k^+ : \Omega_k^+ \to \mathbb{C}^2$ and $\varphi_k^- : \Omega_k^- \to \mathbb{C}^2$ conjugating F with $(z, w) \mapsto (z+1, w)$.

The flower theorem

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$\star$ What if the hypothesis is not satisfied?

Theorem

Let 0 be a non-degenerate singularity of $F \in \mathrm{Diff}(\mathbb{C}^2, 0)$. If $n = 0$, assume that

$$\mathrm{Re}(b/a) < 0;$$

if $n \geq 1$, assume that either

$$\mathrm{Re}\left(\frac{am + bn}{a}\right) < 0 \quad \text{or} \quad \mathrm{Re}\left(\frac{am + bn}{b}\right) < 0.$$

Then there exists a neighborhood $\mathcal{U}$ of the origin such that for every $p \in \mathcal{U}$ outside the fixed set (and outside the parabolic curves if $n = 0$) there exist $j, l \in \mathbb{N}$ such that $F^j(p) \notin \mathcal{U}$ and $F^{-l}(p) \notin \mathcal{U}$.

The flower theorem

⋆ The hypotheses in the previous theorems are necessary:

Example

Let F be the time-1 flow of the vector field

$$X = x^m y^n \left[ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} \right].$$

If

$$\mathrm{Re}\left(\frac{am+bn}{a}\right) = 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{am+bn}{b}\right) \geq 0$$

then for any neighborhood $\mathcal{U}$ of the origin there exists $p \in \mathcal{U}$ outside the fixed set such that the orbit $\{F^j(p) \colon j \in \mathbb{Z}\}$ is contained in $\mathcal{U}$ and bounded away from the origin.

Idea of the proof

We assume for simplicity that $m = 1$ in case $n = 0$ and that $(m, n) = 1$ if $n \geq 1$. With a linear change of coordinates we replace a and b by $-a/(am + bn)$ and $-b/(am + bn)$, so we can directly assume

$$\operatorname{Re} a < 0, \ \operatorname{Re} b < 0 \ \text{ and } am + bn = -1.$$

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In case $n = 0$, Écalle and Hakim showed the existence of parabolic curves for F: there exist a holomorphic injective map u such that $F_2(x, u(x)) = u(F_1(x, u(x)))$. With the sectorial change of coordinates $y \mapsto y - u(x)$ we get

$$F(x, y) = \left(x + x^2 \left[-1 + O(x, y)\right], y + xy \left[b + O(x, y)\right]\right),$$

so we assume that

$$F(x, y) = \left(x + x^{m+1} y^n \left[a + O(x, y)\right], y + x^m y^{n+1} \left[b + O(x, y)\right]\right)$$

with $m \geq 1$, $n \geq 0$, $\operatorname{Re} a < 0$, $\operatorname{Re} b < 0$ and $am + bn = -1$.

Idea of the proof

If $(x_1, y_1) = F(x, y)$ and $n = 0$, then

$$x_1 = x - x^2 + x^2 O(x, y),$$

so for y small x_1 behaves as a tangent to the identity map in dimension 1, and we find an attracting petal bisected by $\mathbb{R}^+$. The domain

$$\mathcal{D} = \left\{(x, y) \in \mathbb{C}^2 : x^m \in V_{\varepsilon,\theta}, |y| < \delta\right\},$$

for some small sector $V_{\varepsilon,\theta}$ bisected by $\mathbb{R}^+$ and some $\delta > 0$, is invariant and attracting.

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If $n \geq 1$, then

$$x_1^m y_1^n = x^m y^n + (x^m y^n)^2(-1 + O(x, y)),$$

so for (x, y) small we find an attracting petal for $x^m y^n \mapsto x_1^m y_1^n$ bisected by $\mathbb{R}^+$. The domain

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for some small sector $V_{\varepsilon,\theta}$ bisected by $\mathbb{R}^+$ and some $\delta > 0$, is invariant and attracting.

Idea of the proof

To get Fatou coordinates, we look for a map $\varphi = (\varphi_1, \varphi_2)$ such that

$$\varphi_1 \circ F = \varphi_1 + 1; \quad \varphi_2 \circ F = \varphi_2.$$

To find φ_2, we use the dynamics of the toy model $\exp X$, where

$$X = x^m y^n \Big(ax \frac{\partial}{\partial x} + by \frac{\partial}{\partial y} \Big).$$

This vector field has first integrals $x^{kb} y^{-ka}$, $k \in \mathbb{C}^*$.
We consider the function

$$g(x, y) = yx^b$$

in case $n = 0$ and

$$g(x, y) = \text{a branch of } x^b y^{-a} \text{ defined in } \mathcal{D}$$

in case $n \geq 1$.

Idea of the proof

The function φ_2 given by

$$\varphi_2(x,y) = \lim_{j\to\infty} g(x_j, y_j),$$

where $(x_j, y_j) = F^j(x,y)$, is well defined and holomorphic in a domain $\mathcal{U} \subset \mathcal{D}$ (which is also invariant and attracting and contains eventually all the convergent orbits of F). And clearly $\varphi_2 \circ F = \varphi_2$.

Now we consider the map $\phi : \mathcal{U} \to \mathbb{C}^2$ defined by

$$\phi(x,y) = \left(\frac{1}{x^m y^n}, \varphi_2(x,y) \right),$$

which is injective and satisfies

$$\phi \circ F \circ \phi^{-1}(z,w) = (z + 1 + h(z,w), w),$$

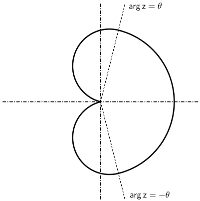
with $h(z,w) = O(z^{-\delta})$, $\delta > 0$. To find φ_1, we use the same techniques as in the 1-dimensional case.

Idea of the proof

To enlarge the parabolic domains, we consider the domain

$$\widetilde{\mathcal{D}} = \left\{ (x,y) \in \mathbb{C}^2 : x^m y^n \in \widetilde{V}_{\varepsilon,\theta}, |nx| < \rho, |y| < \rho \right\},$$

where $\widetilde{V}_{\varepsilon,\theta}$ is



For ρ small, the orbit of any point $(x,y) \in \widetilde{\mathcal{D}}$ eventually lies in $\mathcal{U}$. So if we define

$$\Omega^+ = \bigcup_{j \geq 0} F^j(\widetilde{\mathcal{D}}),$$

which is clearly invariant, we can extend the Fatou coordinate to Ω^+.