A Julia–Wolff–Carathéodory theorem in convex domains of finite type

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Non-tangential limits and Fatou's theorem

Definition (Non-tangential limit)

Let $f\colon \mathbb{D} \to \mathbb{C}$ holomorphic, $\xi \in \partial\mathbb{D}$. $\angle \lim_{z\to\xi} f(z) = L$ iff $f(z_k) \to L$ for every sequence (z_k) converging to ξ non-tangentially (inside a cone).

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Fatou's theorem

$f\colon \mathbb{D} \to \mathbb{D}$ holomorphic admits non-tangential limit at a.e. $\xi \in \partial\mathbb{D}$.

More local information: Julia's lemma

Definition (Dilation and contact point)

$f : \mathbb{D} \to \mathbb{D}$ holomorphic, $\xi \in \partial\mathbb{D}$. The *dilation* λ_ξ is defined by

$$\lambda_\xi = \liminf_{z \to \xi} \frac{1 - |f(z)|}{1 - |z|} \in (0 + \infty].$$

ξ is a *regular contact point if* $\lambda_\xi < +\infty$*.*

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Julia's lemma

If $\xi \in \partial\mathbb{D}$ is regular contact point, then $\exists \eta \in \partial\mathbb{D}$ s.t.

$$\angle \lim_{z \to \xi} f(z) = \eta$$

Proof using horospheres

Definition (Horosphere)

The *horosphere* of radius R centered in ξ is

$$E(\xi, R) := \left\{ z \in \mathbb{D} \colon \frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} < R \right\}.$$

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Proof of Julia's lemma

There exists a point $\eta \in \partial\mathbb{D}$ s.t.

$$f(E(\xi, R)) \subset E(\eta, \lambda R), \quad \forall R > 0.$$

A sequence (z_k) converging to ξ non-tangentially eventually enters every horosphere $E(\xi, R)$.

JWC theorem in the disc

Julia–Wolff–Carathéodory's theorem

If $\xi \in \partial\mathbb{D}$ is regular contact point with non-tg. limit η, then

$$\angle \lim_{z \to \xi} f'(\xi) = \eta \lambda_\xi \bar{\xi}.$$

Koranyi regions in the unit ball

Definition (Koranyi region)

The *Koranyi region* of amplitude $M > 1$ and vertex $\xi \in \mathbb{B}^n$ is

$$K(\xi, M) := \left\{ z \in \mathbb{B}^n \colon \frac{|1 - \langle z, \xi \rangle|}{1 - \|z\|} < M \right\}.$$

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Definition (K-limit)

Let $f \colon \mathbb{B}^n \to \mathbb{C}$ holomorphic, $\xi \in \partial\mathbb{B}^n$, then

$$K\text{-}\lim_{z \to \xi} f(z) = L$$

iff $f(z_n) \to L$ for every sequence (z_n) converging to ξ inside a Koranyi region.

Rudin's JWC in the ball

The Julia lemma in the ball gives the existence of $K\text{-}\lim_{z\to\xi} f(z)$

Rudin's JWC theorem

($q = 2$ for clarity) Let $f\colon \mathbb{B}^2 \to \mathbb{B}^2$ holomorphic. Let $\xi \in \partial\mathbb{B}^2$ be a regular contact point with K-limit η. Write $d_z f$ as

$$\begin{pmatrix} \langle \frac{\partial f}{\partial n_\xi}, n_\eta \rangle & \langle \frac{\partial f}{\partial \tau_\xi}, n_\eta \rangle \\ \langle \frac{\partial f}{\partial n_\xi}, \tau_\eta \rangle & \langle \frac{\partial f}{\partial \tau_\xi}, \tau_\eta \rangle \end{pmatrix}$$

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(4) $\langle \frac{\partial f}{\partial \tau_\xi}, \tau_\eta \rangle$ is bounded in every Koranyi region

Restricted K-limits

Definition (Special restricted sequence)

A sequence $(z_k) \to \xi$ is *special restricted* if it enters a Koranyi region and if

$$\frac{\|z_k - \langle z_k, \xi\rangle \xi\|^2}{1 - |\langle z_k, \xi\rangle|^2} \to 0.$$

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Let $f : \mathbb{B}^n \to \mathbb{C}$ holomorphic, $\xi \in \partial \mathbb{B}^n$.

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It is an intermediate notion between non-tg. limit and K-limit

The proof is based on a Lindelöf principle

Rudin's Lindelöf principle

If f is bounded on every Koranyi region with vertex ξ, and if it admits a limit L on a special restricted curve γ with endpoint ξ, then

$$\angle K\text{-}\lim_{z\to\xi} f(z) = L.$$

Bringing geometry back: D strongly convex domain

Intrinsic definitions (with base-points)

$$\log \lambda_{\xi,p,q} = \liminf_{z\to\xi} k_D(z,p) - k_{D'}(f(z),q)$$

$$E_p(\xi,R) := \{z \colon \lim_{w\to\xi} k_D(z,w) - k_D(w,p) < \log R\}$$

$$K_p(\xi,M) = \{z \colon k_D(z,p) + \lim_{w\to\xi} k_D(z,w) - k_D(w,p) < 2\log M\}$$

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What about special restricted sequences?

Complex geodesics and left inverses

Definition (Complex geodesic)

A holomorphic map $\varphi\colon \mathbb{D} \to D$ is a *complex geodesic* if $k_{\mathbb{D}}(z, w) = k_D(\varphi(z), \varphi(w))$ for all $z, w \in \mathbb{D}$.

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If D is convex, then every complex geodesic admits a left inverse. If D strongly convex, then for all $z \in D$ and $\zeta \in \partial D$ there exists a unique complex geodesic φ s.t. $\varphi(0) = z$ and $\varphi(1) = \zeta$. Moreover $\varphi'(1)$ exists and is transversal to ∂D.

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Definition (Special and restricted sequence)

Fix $p \in D$ and a complex geodesic s.t. $\varphi(0) = p$, $\varphi(1) = \xi$. Choose a left inverse. A sequence (z_k) converging to ξ is *special and restricted* if it is contained in a Koranyi region and if $k_D(z_k, \rho(z_k)) \to 0$.

Abate's JWC theorem

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Why $-\frac{1}{2}$ and $\frac{1}{2}$?

Towards finite type

Definition (Type)

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D convex domain, $\xi \in \partial D$ smooth point. The *(line) type L* at ξ is the maximum order of contact of ∂D with complex lines in ξ. D has finite type L if $L = \max_{\xi \in \partial D} \text{type}(\xi)$.

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D convex domain, $\xi \in \partial D$ smooth point. The *(line) type* L at ξ is the maximum order of contact of ∂D with complex lines in ξ. D has finite type L if $L = \max_{\xi \in \partial D} \mathrm{type}(\xi)$. A strongly convex domain D has type 2.

Definition (Kobayashi type)

D convex domain, $\xi \in \partial D$ smooth point of finite type, $v \in \mathbb{C}^n \setminus \{0\}$. The *Kobayashi type* at ξ in the direction v is

$$s_\xi(v) := \inf\{s \colon d(z, D)^s \kappa_D(z, v) \text{ is bounded on Koranyi regions}\}$$

Kobayashi type in action

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D convex finite type [with technical assumptions].Let $f\colon D \to \mathbb{D}$ holomorphic. Let $\xi \in \partial D$ be a regular contact point with K-limit τ, and let $v \in \mathbb{C}^n \setminus \{0\}$.
Then

$$d(z, \partial D)^{s_\xi(v)-1} \frac{\partial f}{\partial v}$$

is bounded on Koranyi regions.

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Then

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is bounded on Koranyi regions.

Moreover, if $v \notin T^{\mathbb{C}}_\xi \partial D$, then $s_\xi(v) = 1$ and $\frac{\partial f}{\partial v}$ has a nonzero restricted K-limit.

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Moreover, if $v \notin T_\xi^{\mathbb{C}} \partial D$, then $s_\xi(v) = 1$ and $\frac{\partial f}{\partial v}$ has a nonzero restricted K-limit.
If $v \in T_\xi^{\mathbb{C}} \partial D$, then $1/L \leq s_\xi(v) \leq 1 - 1/L$ and

$$d(z, \partial D)^{s_\xi(v)-1} \frac{\partial f}{\partial v} \xrightarrow{\angle K} 0.$$

Idea of proof

Let $z \in K(\xi, M)$, and $\psi : \mathbb{D} \to D$ a complex geodesic such that $\psi(0) = z$ and $\psi'(0) = v/\kappa_D(z, v)$, then

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$$2\pi(1 - \tilde{\rho}(z))^{s-1} \frac{\partial f}{\partial v}(z) = 2\pi(1 - \tilde{\rho}(z))^{s-1} \kappa_D(z, v)(f \circ \psi)'(0) =$$

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$$\int_0^{2\pi}\frac{f(\psi(re^{i\theta}))-\tau}{\tilde{\rho}(\psi(re^{i\theta}))-1}\frac{\tilde{\rho}(\psi(re^{i\theta}))-1}{\tilde{\rho}(z)-1}\left(\frac{\tilde{\rho}(z)-1}{d(z,\partial D)}\right)^s\frac{d(z,\partial D)^s\kappa_D(z,v)}{re^{i\theta}}d\theta.$$

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Definition (Function Ω)

For all $z \in D$ define the function $\Omega_\xi : D \to (-\infty, 0)$ as

$$\Omega_\xi(z) = -\frac{1}{\varphi'_N(1)},$$

where φ is a complex geodesic such that $\varphi(0) = z$ and $\varphi(1) = \xi$.

A pluricomplex Poisson kernel

Horospheres centered in ξ exist (AFGG '22), and

$$\lim_{w\to\xi} k_D(z,w) - k_D(w,p) = \log|\Omega_\xi(p)| - \log|\Omega_\xi(z)|,$$

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$$\lim_{w\to\xi} k_D(z,w) - k_D(w,p) = \log|\Omega_\xi(p)| - \log|\Omega_\xi(z)|,$$

hence the level sets of Ω are exactly the horospheres centered in ξ:

$$E_p(\xi,R) = \{h_{\xi,p} < \log R\} = \{\Omega_\xi < 1/R\}.$$

A pluricomplex Poisson kernel

Horospheres centered in ξ exist (AFGG '22), and

$$\lim_{w\to\xi} k_D(z,w) - k_D(w,p) = \log|\Omega_\xi(p)| - \log|\Omega_\xi(z)|,$$

hence the level sets of Ω are exactly the horospheres centered in ξ:

$$E_p(\xi,R) = \{h_{\xi,p} < \log R\} = \{\Omega_\xi < 1/R\}.$$

$\Omega_1^{\mathbb{D}}(\zeta) = -\frac{1-\|\zeta\|^2}{|1-\zeta|^2}$, and if D is strongly convex Ω coincides with the pluricomplex Poisson Kernel introduced by Bracci-Patrizio-Trapani.

Main result

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(4) $\langle \frac{\partial f}{\partial \tau_\xi}, \tau_\eta \rangle d(z, \partial D)^{\frac{1}{L(v)} - \frac{1}{L'}}$ is bounded in every Koranyi region

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Scaling to show that $L(v) = \frac{1}{s_\xi(v)}$.

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$$\begin{aligned}\lambda_{\xi,p,0}(\tilde{f}) &= \lim_{t\to 1} \exp[k_{\mathbb{D}}(0,t) - k_{\mathbb{D}}(\tilde{f}(\varphi(t)),0)] \\ &= \lim_{t\to 1} \exp[k_{\mathbb{D}}(0,t) - k_{\mathbb{D}}(\tilde{f}(\sigma(t)),0)] = \lim_{t\to 1} \frac{\partial \tilde{f}}{\partial \nu}(\sigma(t)).\end{aligned}$$

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$$= \lim_{t\to 1} \exp[k_{\mathbb{D}}(0,t) - k_{\mathbb{D}}(\tilde{f}(\sigma(t)),0)] = \lim_{t\to 1} \frac{\partial \tilde{f}}{\partial \nu}(\sigma(t)).$$

Hence

$$\angle K\text{-}\lim_{z\to\xi} \frac{\partial \tilde{f}}{\partial n_\xi}(z) = \lambda_{\xi,p,0}(\tilde{f})|\Omega_\xi^D(p)| = \lambda_{\xi,p,q}(f)|\Omega_\xi^D(p)|.$$

Moreover $\angle K\text{-}\lim_{w\to\eta} \frac{\partial \tilde{\rho}_q}{\partial n_\eta}(w) = |\Omega_\eta^{D'}(q)|$.