Classification of reversible parabolic diffeomorphisms of $(\mathbb{C}^2, 0)$ and of exceptional hyperbolic CR-singularities

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(joint work with Martin Klimeš)

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Motivation [Bishop '65], [Moser–Webster '83]

Analytic real surface $M \subset \mathbb{C}^2$

$$M: \quad z_2 = \begin{cases} \gamma^{-1} z_1 \bar{z}_1 + z_1^2 + \bar{z}_1^2 + \mathrm{h.o.t.}, & 0 < \gamma \leq \infty \\ z_1 \bar{z}_1 + \mathrm{h.o.t.}, & \gamma = 0. \end{cases}$$

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CR-singularity at 0, isolated if $\gamma \neq \frac{1}{2}$:

$$\begin{cases} z = 0 : & T_0 M = \{z_2 = 0\} \\ z \neq 0 : & T_z M \cap i T_z M = \{0\} \text{ (totally real).} \end{cases}$$

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Problem: Classification of $(M, 0)$ by *biholomorphic transformations*

$$z \mapsto \psi(z), \quad \bar{z} \mapsto \overline{\psi(z)}, \qquad \psi \in \mathrm{Diff}(\mathbb{C}^2, 0).$$

Real surface in $\mathbb{C}^2$

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Anti-holomorphic involution $\rho : (z, w) \mapsto (\overline{w}, \bar{z})$,

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Transformations: $(z, w) \mapsto \big(\psi(z), \overline{\psi}(w)\big), \qquad \psi \in \mathrm{Diff}(\mathbb{C}^2, 0).$

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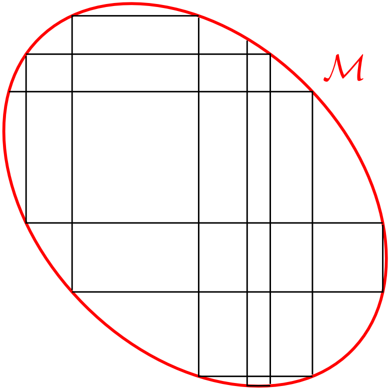

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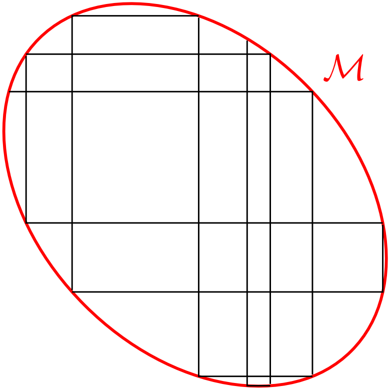

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Coordinates (z_1, w_1) on $(\mathcal{M}, 0)$:

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Theorem [Moser–Webster]

Isomorphism:

$$\left\{ \begin{matrix} \textit{Triples } (\tau_1, \tau_2, \rho) \\ \textit{with } \lambda, \lambda^{-1} \end{matrix} \right\} \Big/ _{\mathrm{Diff}(\mathbb{C}^2,0)} \quad \longleftrightarrow \quad \left\{ \begin{matrix} \textit{Surfaces } (M, 0) \\ \textit{with } \gamma \end{matrix} \right\} \Big/ _{\mathrm{Diff}(\mathbb{C}^2,0)}$$

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Under this correspondence:

$$\begin{array}{c} \phi \textit{ has first integral} \\ \textit{of Morse type} \\ H = H \circ \phi \end{array} \longleftrightarrow \begin{array}{c} M \textit{ is holomorphically flat} \\ M \subset \textit{Levi flat hypersurface} \\ \{\mathrm{Im}\, z_2 = 0\} \end{array}$$

$M = \{z_2 = \bar{z}_2 = H(z_1, \bar{z}_1)\} \implies H(z_1, w_1)$ is Morse first integral.

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[Moser–Webster '83] *Analytic equivalence to normal form*

$$\tau_{1,\mathrm{nf}} = \phi_{\mathrm{nf}}^{\circ(\frac{1}{2})} \circ \sigma, \quad \tau_{2,\mathrm{nf}} = \sigma\phi_{\mathrm{nf}}^{\circ(\frac{1}{2})}, \quad \rho_{\mathrm{nf}} = \sigma\bar{\xi}, \quad \sigma = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),$$

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$$M_{\mathrm{nf}}: \; z_2 = \left(\lambda^{\frac{1}{2}} e^{\epsilon\,(-z_2)^s} + \lambda^{-\frac{1}{2}} e^{-\epsilon\,(-z_2)^s}\right) z_1 \bar{z}_1 + z_1^2 + \bar{z}_1^2,$$

holomorphically flat.

$\rightsquigarrow$ higher degeneracy/dimension [Gong–S. '16,'19]

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Small divisors: KAM-type results (invariant curves), or topological obstructions to convergence.

[Gong ’94,’96,’04], [S.–Zhao ’22]

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V) *Exceptional case* $\lambda^p = 1$, $p \geq 2$:
under assumption of Morse 1st integral [Klimeš–S. '22]

Theorem [Klimeš–S.]

(τ_1, τ_2, ρ), $(\tau_1', \tau_2', \rho')$ *with Morse first integral,* $\lambda^p = 1$, $p \geq 2$, *are conjugated by* $\mathrm{Diff}_{\mathrm{id}}(\mathbb{C}^2, 0)$ *if and only if* (τ_1, τ_2), (τ_1', τ_2') *are.*

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Embedding of the moduli space

$$\left\{ \begin{array}{c} \textit{hol. flat } M \\ \textit{with} \\ \textit{exceptional } \gamma \end{array} \right\} \Big/ _{\mathrm{Diff}_{\mathrm{id}}(\mathbb{C}^2,0)} \hookrightarrow \left\{ \begin{array}{c} (\tau_1, \tau_2) \textit{ with} \\ \lambda^p = 1,\ p \geq 2, \\ \textit{\& Morse 1st integral} \end{array} \right\} \Big/ _{\mathrm{Diff}_{\mathrm{id}}(\mathbb{C}^2,0)} .$$

Image described by some explicit antiholomorphic symmetry.

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has a unique *formal infinitesimal generator* $\hat{X}$:

$$\phi^{\circ p} = \exp(\hat{X}), \quad \tau^*\hat{X} = -\hat{X}, \quad \hat{X}.H = 0.$$

Theorem: Formal classification [Klimeš–S.]

(ϕ, τ, H) is formally conjugated by $\widehat{\mathrm{Diff}}_{\mathrm{id}}(\mathbb{C}^2, 0)$ to

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- $h = \xi_1\xi_2$, $u = \xi_1^p + \xi_2^p$ *are basic* $\langle\sigma, \Lambda\rangle$*-invariants,*
- $P(u,h) = u^k + P_{k-1}(h)u^{k-1} + \ldots + P_0(h)$ *analytic,* $P(u,0) = u^k$,
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This normal form is unique up to action $\xi \mapsto a\xi$, $a \in \mathbb{C}^*$.

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(o) $\phi_{\rm nf}(\xi) = \Lambda\xi, \quad \tau_{\rm nf}(\xi) = \sigma\xi.$

The conjugacy is *always convergent*: $\mathcal{G}_{\rm nf} = \langle \phi_{\rm nf}, \tau_{\rm nf} \rangle$ is finite & linear.

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(a) $\phi_{\mathrm{nf}}(\xi) = \begin{pmatrix} \lambda e^{\frac{c}{p}h^s} & 0 \\ 0 & \lambda^{-1} e^{-\frac{c}{p}h^s} \end{pmatrix} \xi, \quad \tau_{\mathrm{nf}}(\xi) = \sigma\xi.$

In general, the conjugacy is *divergent for topological reasons*:

if $\frac{c}{p}h^s \in i\pi\mathbb{Q} \implies$ whole leaf $\{h = \mathrm{const}\}$ is periodic.

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The conjugacy is *always convergent*: $\mathcal{G}_{\rm nf} = \langle \phi_{\rm nf}, \tau_{\rm nf} \rangle$ is finite & linear.

(a) $\phi_{\rm nf}(\xi) = \begin{pmatrix} \lambda e^{\frac{c}{p}h^s} & 0 \\ 0 & \lambda^{-1}e^{-\frac{c}{p}h^s} \end{pmatrix} \xi, \quad \tau_{\rm nf}(\xi) = \sigma\xi.$

In general, the conjugacy is *divergent for topological reasons*:

if $\frac{c}{p}h^s \in i\pi\mathbb{Q} \implies$ whole leaf $\{h = \text{const}\}$ is periodic.

(b) $\hat{\phi}_{\rm nf}(\xi) = \Lambda \exp\left(\frac{1}{p}\frac{c\,h^s P(u,h)}{1+c\,P(u,h)\,\hat{\mu}(h)}\left(\xi_1\frac{\partial}{\partial\xi_1} - \xi_2\frac{\partial}{\partial\xi_2}\right)\right)(\xi),$
$\tau_{\rm nf}(\xi) = \sigma\xi.$

In general, the conjugacy is *divergent for analytic reasons*:

realizable "sectorially".

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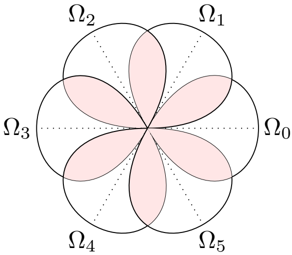
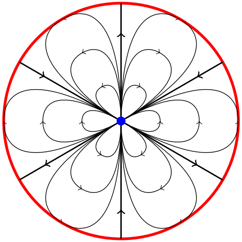
Theorem [Birkhoff '39, Kimura '71]
ϕ conjugated to ϕ_{nf} by a cochain of bounded analytic transformations on a covering by $2kp$ petals $\{\Omega_j\}_{j \in \mathbb{Z}_{2kp}}$,

$$\Psi_{\Omega_j}(z) = z + \mathrm{h.o.t.}, \qquad \Psi_{\lambda\Omega_j} \circ \phi = \phi_{\mathrm{nf}} \circ \Psi_{\Omega_j}.$$

The petals Ω_j are spanned by *real-time trajectories* of $e^{i\theta} X_{\mathrm{nf}}$, $\theta \in]0+, \pi-[$.





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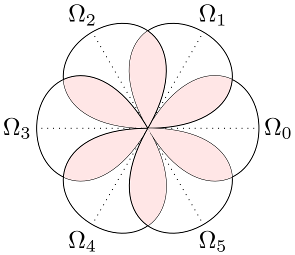

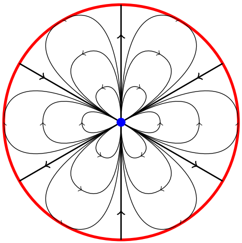

Transition maps $\psi_j = \Psi_{\Omega_{j-1}} \circ \Psi_{\Omega_j}^{\circ(-1)}$ on *intersections* $\Omega_{j-1} \cap \Omega_j$.

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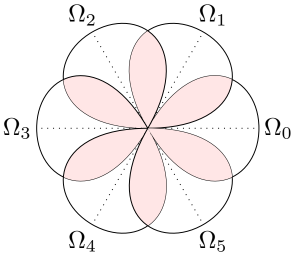

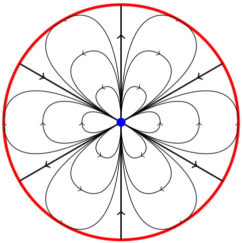

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Theorem [Birkhoff '39, Écalle '75, Voronin '81] *Two formally tangent-to-identity equivalent germs ϕ, ϕ', are analytically tangent-to-identity equivalent if and only if their cocycles $\{\psi_j\}_{j\in\mathbb{Z}_{2kp}}$, $\{\psi'_j\}_{j\in\mathbb{Z}_{2kp}}$ agree*.

The formal case (b)

Reversible diffeo (ϕ, τ) of $(\mathbb{C}^2, 0)$ with Morse first integral $H(\xi)$

Formal normal form:

$$\hat{\phi}_{\mathrm{nf}} = \Lambda \exp\left(\tfrac{1}{p}\hat{X}_{\mathrm{nf}}\right),$$

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Holomorphic model:

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Theorem: "Sectorial" classification [Klimeš–S.]

(ϕ, τ, H) *with model* $(\phi_{\mathrm{mod}}, \sigma, h)$.

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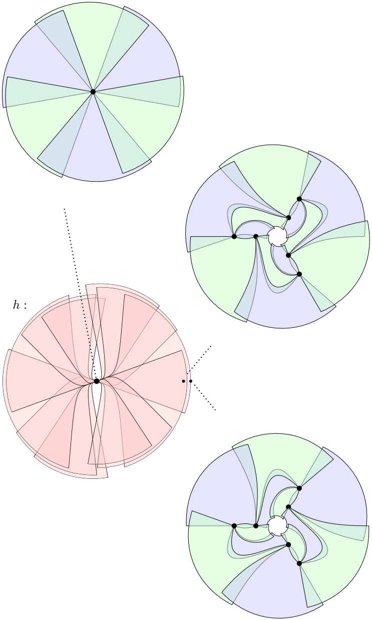

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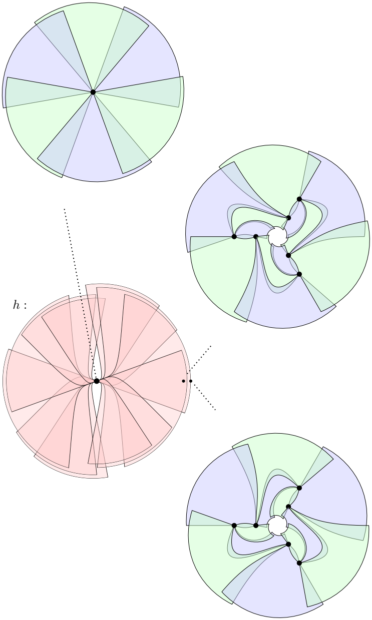
On each S *is:*

covering by $2kp$ *"outer domains" and* $2kp$ *"inner domains"* $\{\Omega_S^j\}_{j \in \mathbb{Z}_{4kp}}$,

cochain of bounded analytic transformations $\Psi_{\Omega_S^j} = \mathrm{id} + \mathrm{h.o.t.}$

$$\Psi_{\Lambda\Omega_S^j} \circ \phi = \phi_{\mathrm{mod}} \circ \Psi_{\Omega_S^j}, \quad \Psi_{\sigma\Omega_S^j} \circ \tau = \sigma \circ \Psi_{\Omega_S^j}.$$



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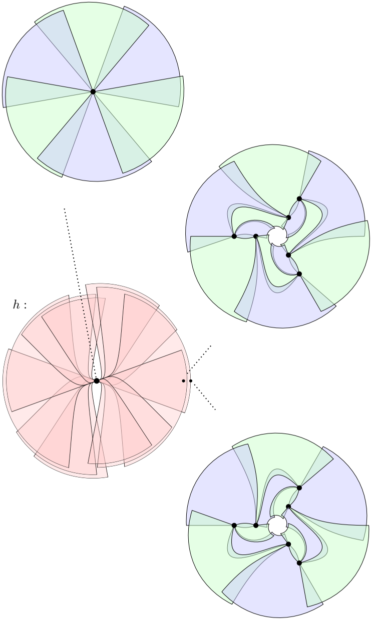
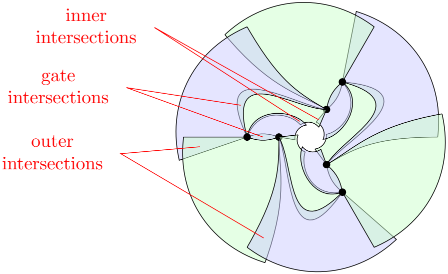
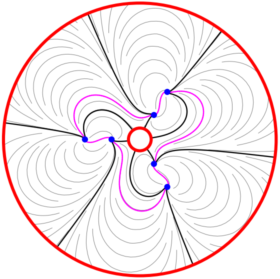
The normalizing cochain is unique up to composition with a flow cochain $\left\{\exp(C_{\Omega_S^j}(h) X_{\mathrm{mod}})\right\}_{j \in \mathbb{Z}_{4kp}}$.



Ω_S^j spanned by the real-time trajectories of $e^{i\theta}\boldsymbol{X}_{\mathrm{mod}}$, $\theta \in]0+,\pi-[$.





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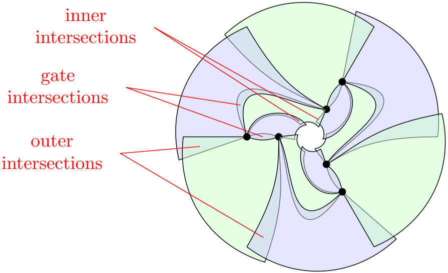

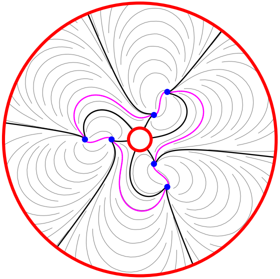

Transition maps $\psi_{S,j} = \Psi_{\Omega_S^{j+1}} \circ \Psi_{\Omega_S^j}^{\circ(-1)}$ on *outer intersections*.

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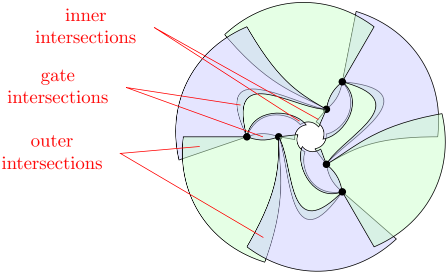

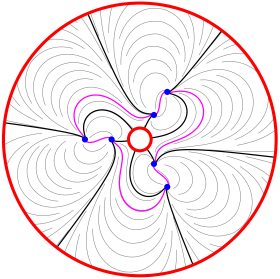

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Outer cocycle on S = equivalence class of $\{\psi_{S,j}\}_{j\in\mathbb{Z}_{2kp}}$ modulo conjugation by flow cochains $\left\{\exp(C_{\Omega_S^j}(h)h^{-s}X_{\mathrm{mod}})\right\}_{j\in\mathbb{Z}_{2kp}}$

$$\psi_{S,j} \simeq \exp(C_{\Omega_S^{j-1}}(h)h^{-s}X_{\mathrm{mod}}) \circ \psi_{S,j} \circ \exp(-C_{\Omega_S^j}(h)h^{-s}X_{\mathrm{mod}})$$

Theorem: Analytic classification [Klimeš–S.]

The following is equivalent:

1. *Two germs* (ϕ, τ), (ϕ', τ') *in the same model class are analytically tangent-to-identity equivalent.*
2. *For every cuspidal sector* S *the associated outer cocycles* $\{\psi_{S,j}\}_{j\in\mathbb{Z}_{2kp}}$, $\{\psi'_{S,j+r}\}_{j\in\mathbb{Z}_{2kp}}$ *agree.*
3. *For one cuspidal sector* S *the associated outer cocycles* $\{\psi_{S,j}\}_{j\in\mathbb{Z}_{2kp}}$, $\{\psi'_{S,j+r}\}_{j\in\mathbb{Z}_{2kp}}$ *agree.*