Formal principle with convergence for rational curves

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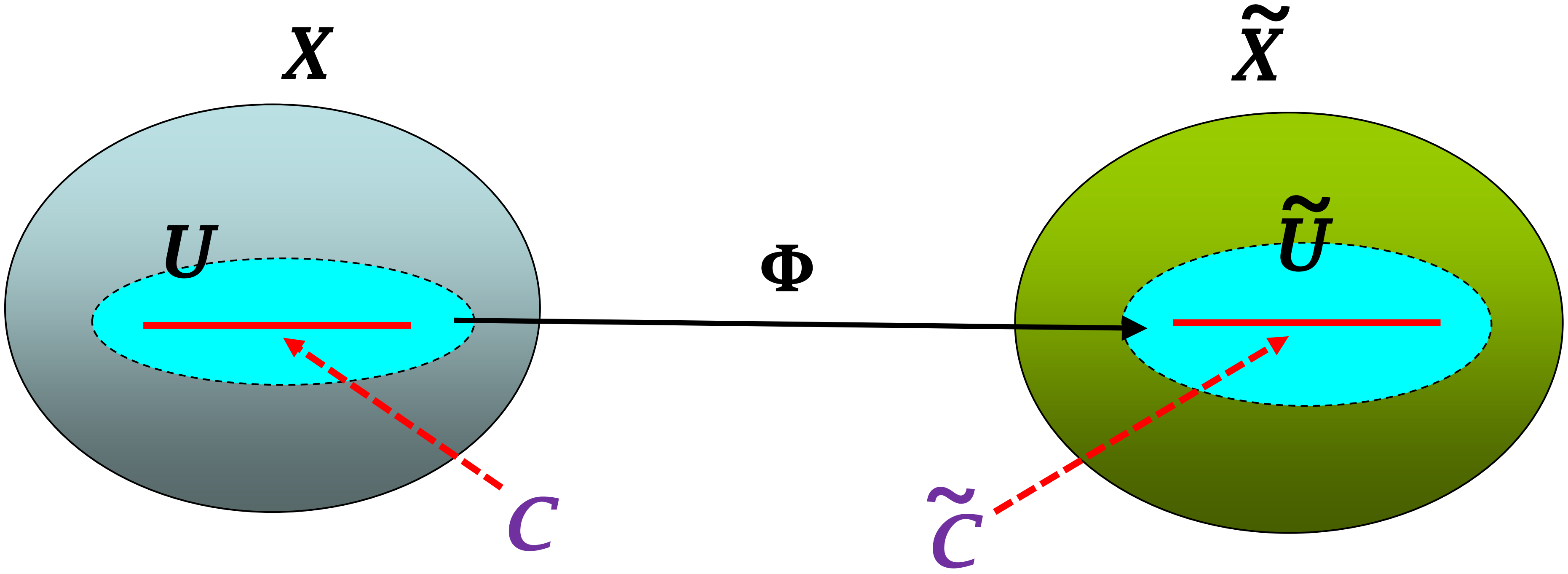
Germs of neighborhoods of submanifolds

- Throughout, let $C \subset X$ (also $\widetilde{C} \subset \widetilde{X}$) be a compact complex submanifold in a (usually noncompact) complex manifold.
- Denote by $(C/X)_{\mathcal{O}}$ the germ of analytic neighborhoods of C in X.
- A biholomorphic map $\Phi : (C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$ means a biholomorphic map $\Phi : U \cong \widetilde{U}$ between some neighborhoods

$$\begin{array}{ccc} C & & \widetilde{C} \\ \cap & & \cap \\ U & \xrightarrow{\Phi} & \widetilde{U} \\ \cap & & \cap \\ X & & \widetilde{X} \end{array}$$

such that $\Phi|_C : C \cong \widehat{C}$.



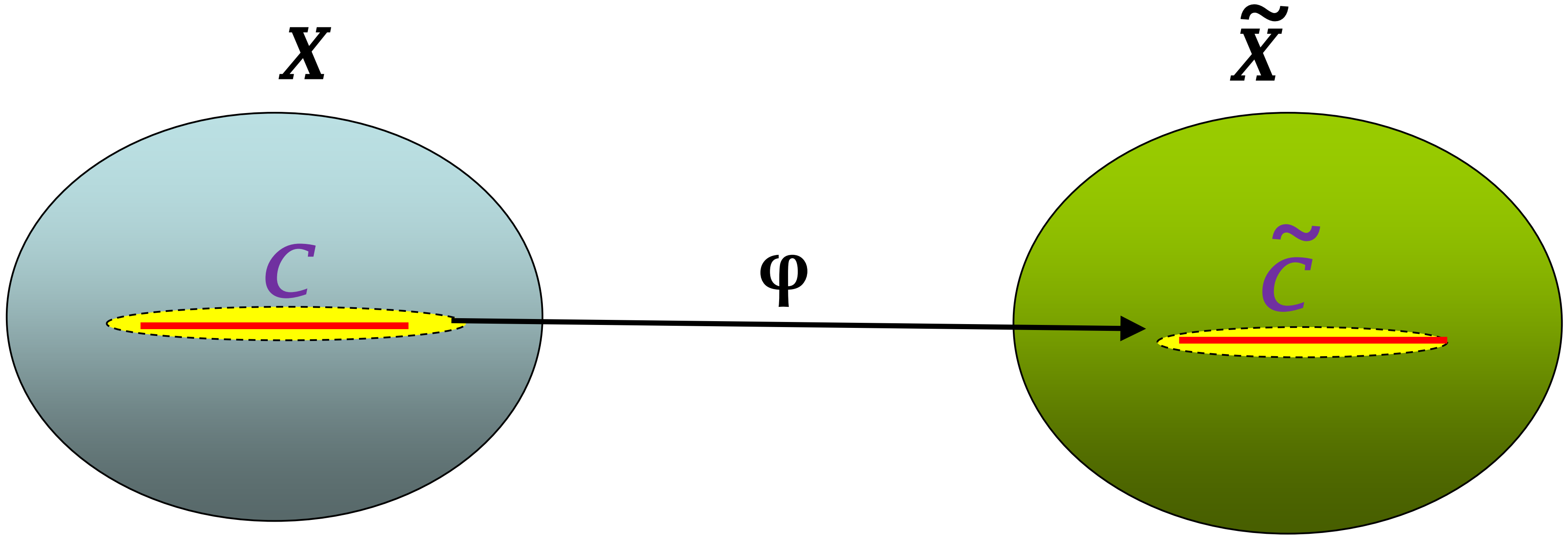
X
$\tilde{X}$
U
$\tilde{U}$
Φ
C
$\tilde{C}$

Formal neighborhoods of submanifolds

- Let $\mathcal{I}_C$ be the ideal sheaf of $C \subset X$ and $(C/X)_k$ be the k-th infinitesimal neighborhood of C in X, namely, the complex space with the structure sheaf $\mathcal{O}_X/\mathcal{I}_C^{k+1}$.
- The collection $(C/X)_\infty$ of $(C/X)_k$ for all $k \geq 0$ is the formal neighborhood of C in X.
- A formal isomorphism $\varphi : (C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_\infty$ means a compatible collection of biholomorphisms of complex spaces

$$\begin{array}{ccc} (C/X)_k & \xrightarrow{\varphi_k} & (\widetilde{C}/\widetilde{X})_k \\ \cap & & \cap \\ (C/X)_{k+1} & \xrightarrow{\varphi_{k+1}} & (\widetilde{C}/\widetilde{X})_{k+1} \end{array}$$

for all $k \geq 0$.



X
X̃
C
C̃
φ

Our Problem

- A biholomorphic map $\Phi : (C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$ of germs induces a formal isomorphism

$$\Phi_\infty := \Phi|_{(C/X)_\infty} : (C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_{infty}.$$

Problem

When does the converse hold?

(i) *If $(C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_\infty$, then is $(C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$?*

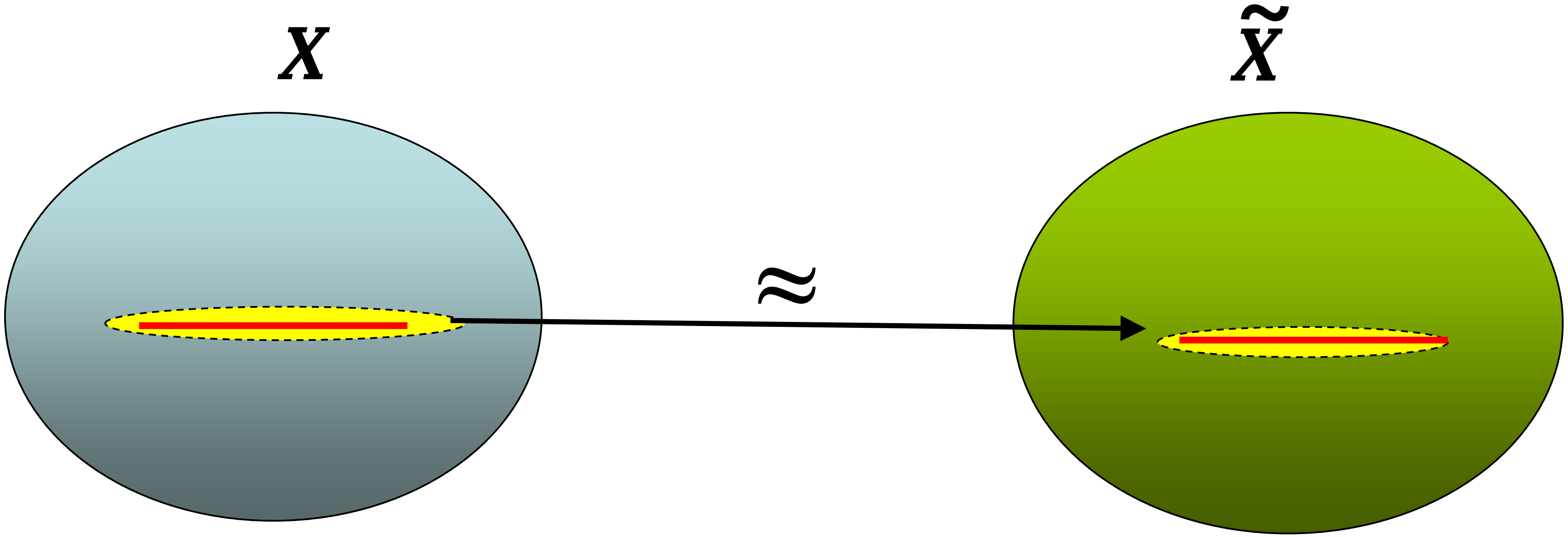
(ii) *Given a formal isomorphism $\varphi : (C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_\infty$, can we find $\Phi : (C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$ such that $\varphi = \Phi_\infty$?*

Formal Principle/Formal Principle with Convergence

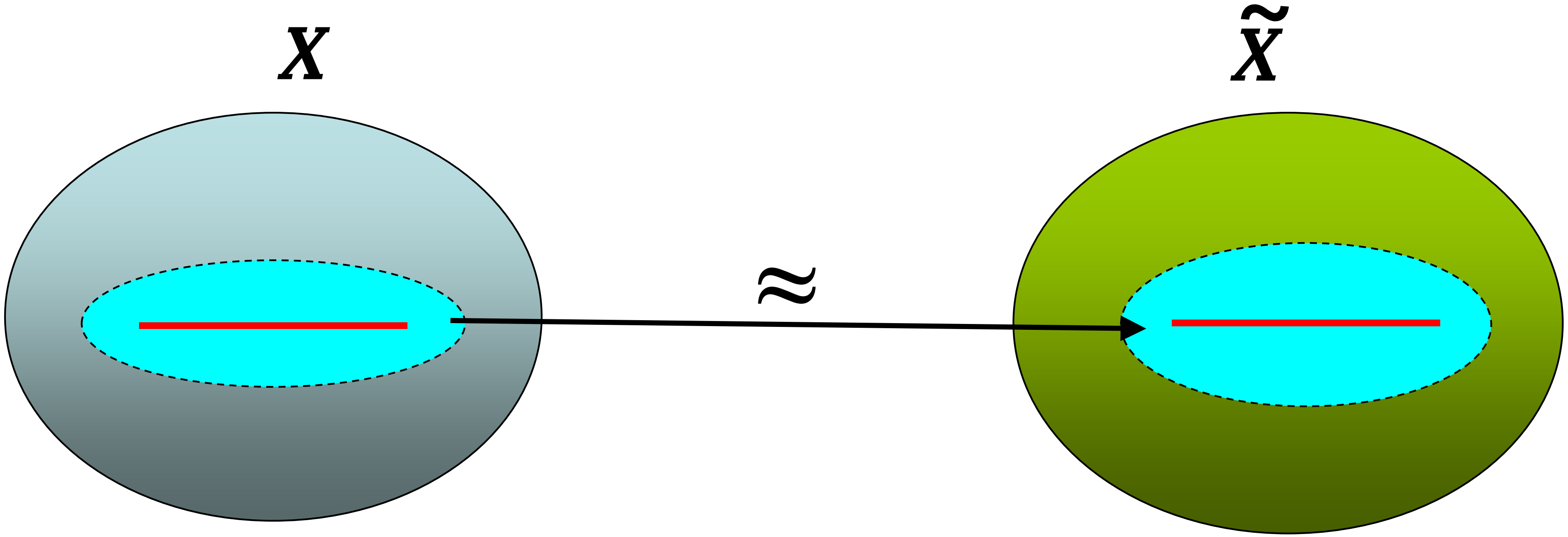
Definition

- We say that $C \subset X$ satisfies the formal principle if the **existence** of a formal isomorphism $(C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_\infty$ implies the **existence** of a biholomorphic map of germs $(C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$.
- We say that $C \subset X$ satisfies the formal principle with convergence if any formal isomorphism $\varphi : (C/X)_\infty \cong (\widetilde{C}/\widetilde{X})_\infty$ comes from a biholomorphic map of germs $\Phi : (C/X)_{\mathcal{O}} \cong (\widetilde{C}/\widetilde{X})_{\mathcal{O}}$, namely, $\varphi = \Phi_\infty$. In other words, **any formal isomorphism is convergent.**

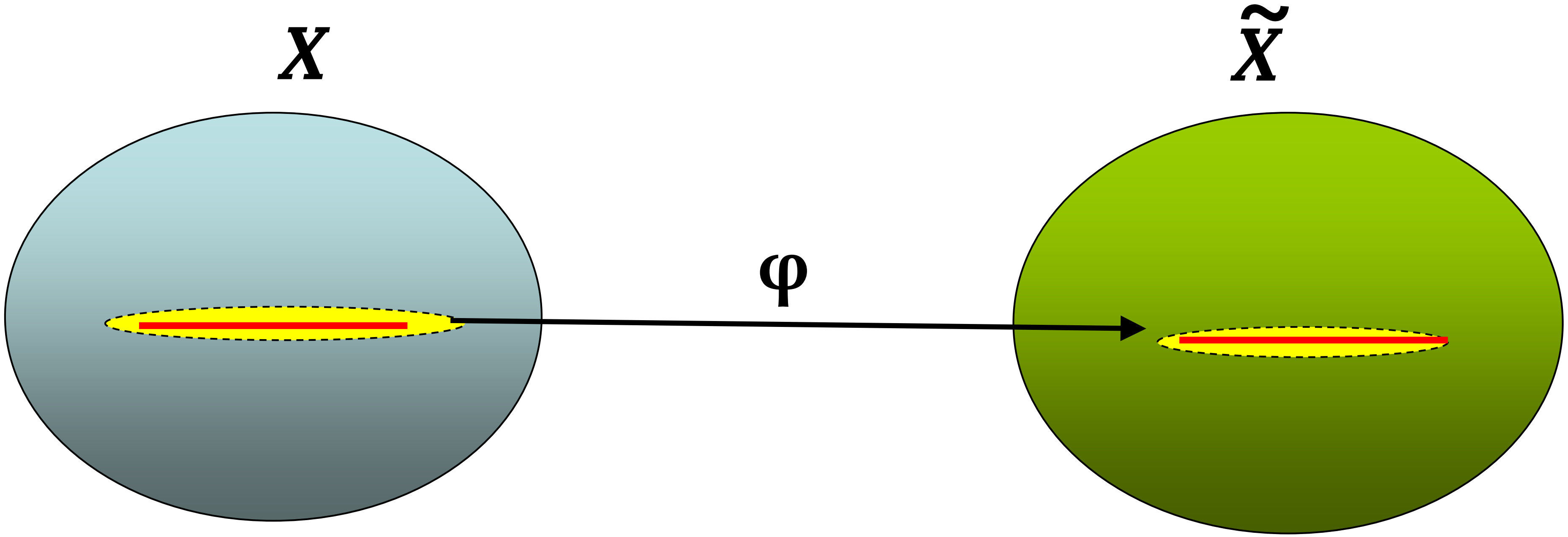
Formal Principle



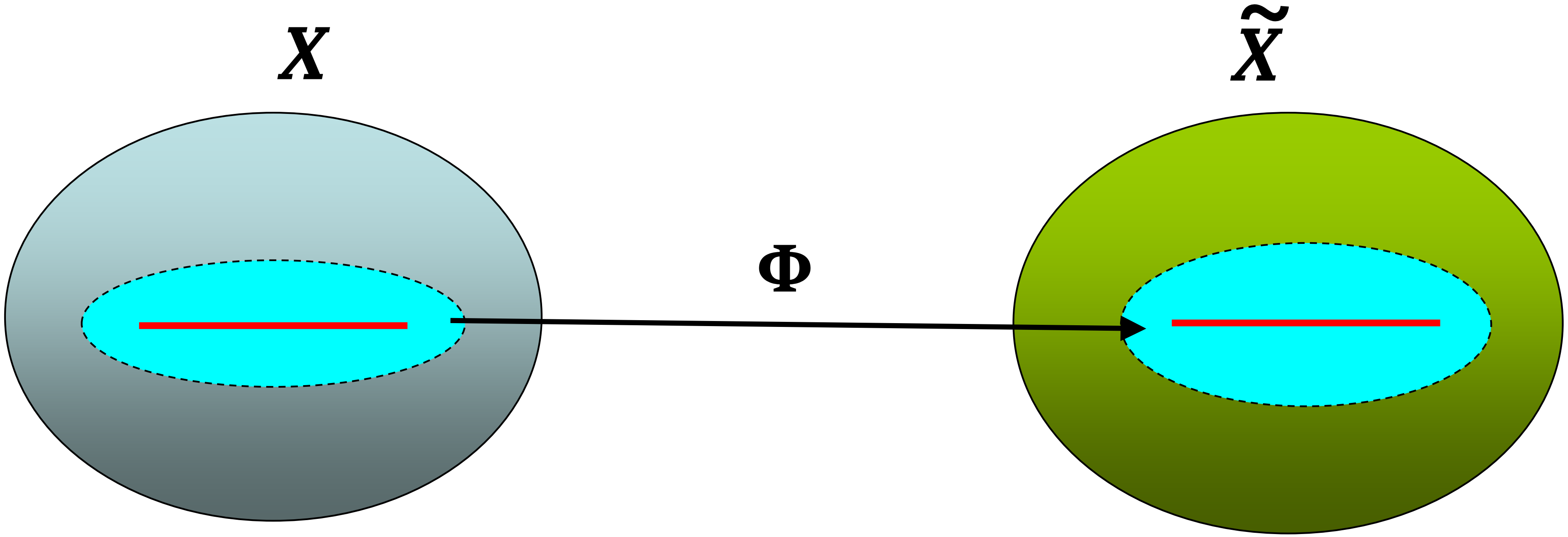
Formal Principle



Formal Principle with Convergence



Formal Principle with Convergence



Example: a point in $\mathbb{C}$

- Let $P \in \mathbb{C}$ be a point with a local coordinate z and let $\widetilde{P} \in \mathbb{C}$ be a point with a local coordinate w.
- Any formal power series
$$w = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$
with $a_1 \neq 0$ gives a formal isomorphism $\varphi : (P/\mathbb{C})_\infty \cong (\widetilde{P}/\mathbb{C})_\infty$.
- φ comes from a biholomorphism of germs $\Phi : (P/\mathbb{C})_\mathcal{O} \cong (\widetilde{P}/\mathbb{C})_\mathcal{O}$ if and only if the formal power series converges.
- Thus $P \in \mathbb{C}$ **satisfies the formal principle**, but does NOT satisfy the **formal principle with convergence**.

Counterexample to the formal principle

Example (V. I. Arnold 1976)

There exists an elliptic curve $C \subset X$ in a complex surface for which the formal principle does NOT hold. Its normal bundle $N_{C/X}$ is topologically trivial.

- All examples of submanifolds violating the formal principle which we know so far are of the nature similar to Arnold's example.
- No simply-connected example violating the formal principle is known.
- In particular, no smooth rational curve $\mathbb{P}^1 \subset X$ violating the formal principle is known.

Vector bundles on rational curves

From now on, we concentrate our discussion to the simplest situation when C and $\widetilde{C}$ are smooth rational curves, namely, biholomorphic to the Riemann sphere $\mathbb{P}^1$. Even in this case, our Problem is highly interesting and difficult.

Definition

A vector bundle V on $\mathbb{P}^1$ can be written as a direct sum

$$V \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r)$$

for some integers $a_1, a_2, \ldots a_r$.

(i) V is a positive vector bundle (denoted by $V > 0$) if $a_1, a_2, \ldots, a_r > 0$.

(ii) V is a semipositive vector bundle (denoted by $V \geq 0$) if $a_1, a_2, \ldots, a_r \geq 0$, equivalently if $V \cong V^+ \oplus \mathcal{O}^q$ for some positive vector bundle V^+ and a trivial vector bundle $\mathcal{O}^q$ with $q = \mathrm{rank}(V) - \mathrm{rank}(V^+)$.

Previous results

Theorem (Grauert, 1962)

Let $N_{C/X}$ be the normal bundle of a smooth rational curve $C \subset X$. If $N_{C/X} < 0$, namely, the dual $N^*_{C/X} > 0$, then $C \subset X$ satisfies the formal principle.

In general, $C \subset X$ in the above theorem does not satisfy the formal principle with convergence.

Theorem (Hirschowitz 1981)

If $N_{C/X} > 0$, then $C \subset X$ satisfies the **formal principle with convergence**.

Conjecture [Hirschowitz 1981] If $N_{C/X} \geq 0$, then $C \subset X$ satisfies the **formal principle**.

Result on Hirschowtiz's conjecture

Conjecture [Hirschowitz 1981] Let $N_{C/X}$ be the normal bundle of a smooth rational curve $C \subset X$. If $N_{C/X} \geq 0$, then $C \subset X$ satisfies the **formal principle**.

Theorem (H. 2019)

If $N_{C/X} \geq 0$, then a general deformation of C in X satisfies the formal principle.

We can NOT strengthen the last statement to "a general deformation of C in X satisfies the **formal principle with convergence**."

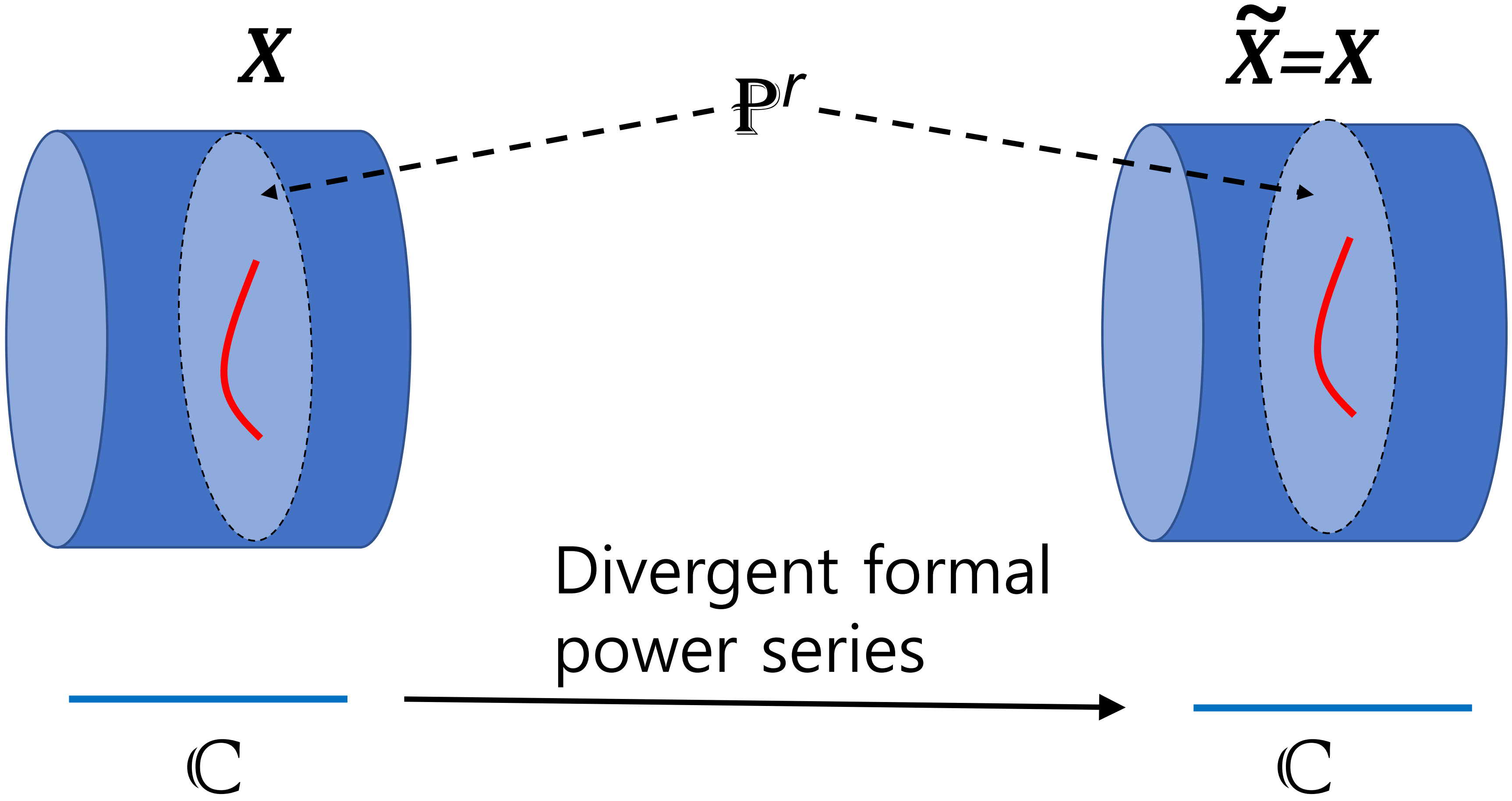
Example: product with $\mathbb{C}$

- Let $C_0 \subset \mathbb{P}^r$ be a smooth rational curve in projective space.
- Set $X \cong \mathbb{P}^r \times \mathbb{C}$ and let $C \subset X$ be

$$C_0 \subset \mathbb{P}^r = \mathbb{P}^r \times 0 \subset \mathbb{P}^r \times \mathbb{C} = X.$$

- The normal bundle $N_{C/S} \cong N_{C_0/\mathbb{P}^r} \oplus \mathcal{O}$ is semipositive.
- Any deformations of C in X lies in a $\mathbb{P}^r$-factor of $X \cong \mathbb{P}^r \times \mathbb{C}$.
- Since a point $P \in \mathbb{C}$ does not satisfy the formal principle with convergence, we can see that no deformation of $C \subset X$ satisfies the formal principle with convergence.

Example : the product $\mathbb{P}^r \times \mathbb{C} = \boldsymbol{X}$



Distribution on a complex manifold

Let M be a complex manifold.

- A distribution D on M means a saturated subsheaf $D \subset TM$ of the tangent sheaf.
- Associated with D are subsheaves

$$D \subset \partial^1 D \subset \partial^2 D \subset \cdots \subset \partial^\ell D \subset TM$$

  such that $\partial^1 D$ is the saturation of $[D, D]$ and $\partial^{i+1} D$ is the saturation of $[\partial^i D, \partial^i D]$ for each $1 \leq i < \ell$.
- By Frobenius's Theorem, D is a foliation if and only if $D = \partial^1 D$.
- We say that D is **bracket-generating** if $\partial^\ell D = TM$ for some $\ell \geq 0$.

Space of deformations of rational curves

- Let $C \subset X$ be a smooth rational curve with $N_{C/X} \geq 0$.
- Let $\mathcal{M}$ be the **space of smooth deformations** $C_t \subset X$ of $C \subset X$ with $N_{C_t/X} \geq 0$.
- $\mathcal{M}$ is a complex manifold and the tangent space at $[C_t] \in \mathcal{M}$ can be identified with $T_{[C_t]}\mathcal{M} = H^0(C_t, N_{C_t/X})$.
- From $N_{C_t/X} \cong N^+_{C_t/X} \oplus \mathcal{O}^q$, we have a distinguished subspace

$$\mathcal{D}_{[C_t]} := H^0(C_t, N^+_{C_t/X}) \subset H^0(C_t, N_{C_t/X}) = T_{[C_t]}\mathcal{M}.$$

- The distribution $\mathcal{D} \subset \mathcal{M}$ is the canonical distribution on $\mathcal{M}$.

Examples of the canonical distributions

Example

When $X = \mathbb{P}^r \times \mathbb{C}$ and $C = C_0 \times 0$, the space $\mathcal{M}$ of deformations of C is naturally isomorphic to $\mathcal{M}_0 \times \mathbb{C}$ where $\mathcal{M}_0$ is the space of deformations of C_0 in $\mathbb{P}^r$. The canonical deformation $\mathcal{D} \subset T\mathcal{M}$ is a **foliation** whose leaves are the $\mathcal{M}_0$-factors of $\mathcal{M} \cong \mathcal{M}_0 \times \mathbb{C}$.

Example

Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface of degree less than n. Let $\mathcal{M}$ be the space of lines lying on X with semipositive normal bundles. Then one can show that the canonical distribution $\mathcal{D} \subset T\mathcal{M}$ is **bracket-generating**.

Formulation of a new conjecture

- Let $\mathcal{M}$ be an irreducible component of the space of smooth rational curves on a complex manifold with semipositive normal bundle.
- Our result in 2019 says that a general member of $\mathcal{M}$ satisfies the formal principle.

New Conjecture If the canonical distribution $\mathcal{D} \subset T\mathcal{M}$ is bracket-generating, then a general member of $\mathcal{M}$ satisfies the formal principle with convergence.

- If $\mathcal{D} = T\mathcal{M}$, this follows from Hirschowitz 1981.

Goursat distribution

Definition

A distribution $D \subset TM$ on a complex manifold is a Goursat distribution if

$$\mathrm{rank}(D) = 2 \text{ and } \mathrm{rank}(\partial^i D) = i + 2$$

for all $1 \leq i \leq \dim M - 2$.

A Goursat distribution is a bracket-generating distribution with the **slowest possible growth** of the successive brackets.

Main Theorem

Theorem (H. 2022)

Let $\mathcal{M}$ be an irreducible component of the space of smooth rational curves with semipositive normal bundles on a complex manifold X. Assume that the canonical distribution $\mathcal{D} \subset T\mathcal{M}$ is a Goursat distribution. Then a general member of $\mathcal{M}$ satisfies the **formal principle with convergence***.*

- When $\mathcal{D}$ is a Goursat distribution, a general member $C \subset X$ of $\mathcal{M}$ satisfies $N_{C/X} \cong \mathcal{O}(1) \oplus \mathcal{O}^q$.
- Thus the normal bundle has the minimal amount of positivity and the growth of the brackets of $H^0(C, N^+_{C/X})$ is the slowest possible.
- Thus the above Theorem verifies our conjecture for the case when the normal bundle is furthest from being positive. This is a good evidence for the conjecture.

Examples when $\mathcal{D} \subset T\mathcal{M}$ is Goursat

Example

When $\dim \mathcal{M} \leq 4$, any bracket generating $\mathcal{D} \subset T\mathcal{M}$ is a Goursat distribution. For example, when $X \subset \mathbb{P}^5$ is a 4-dimensional smooth cubic hypersurface, the canonical distribution on the space of general lines on X is Goursat. Main Theorem says that **a general line on a cubic fourfold satisfies the formal principle with convergence.**

Example

Let $Z \subset \mathbb{P}^{n-1}$ be a nondegenerate smooth curve. Regarding $\mathbb{P}^{n-1}$ as a hyperplane in $\mathbb{P}^n$, let X be the blowup of $\mathbb{P}^n$ with Z as the blowup center. Let $\mathcal{M}$ be the space of proper transformations of lines on $\mathbb{P}^n$ intersecting Z and not contained in $\mathbb{P}^{n-1}$. Then $\mathcal{D} \subset T\mathcal{M}$ is a Goursat distribution, providing many examples of Main Theorem in any dimension.

Geometric feature of Proof of Main Theorem

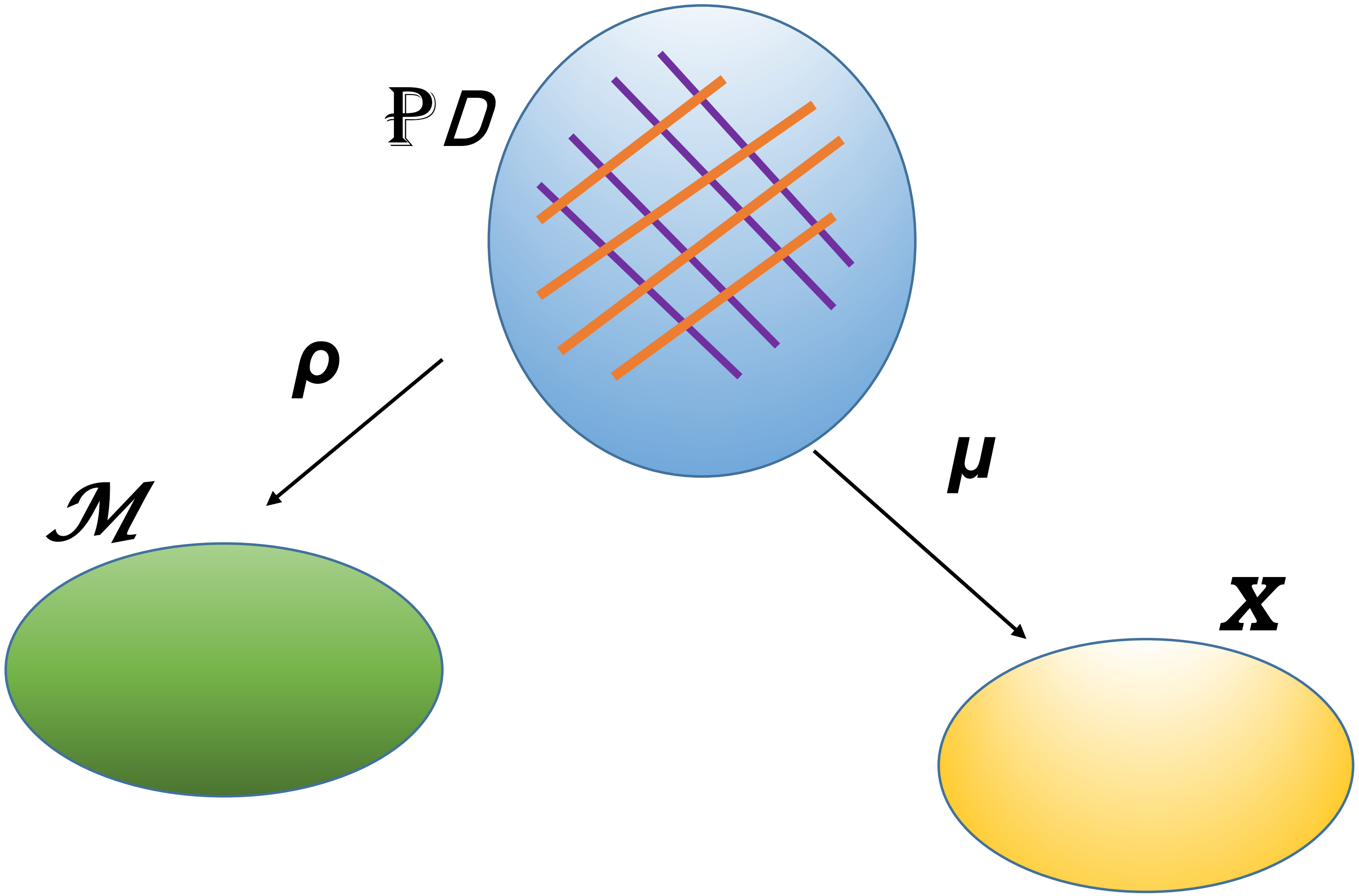
- When $\mathcal{D} \subset T\mathcal{M}$ is Goursat, each nonzero element $s \in \mathcal{D}_{[C]}$ at a point $[C] \in \mathcal{M}$ corresponding to a rational curve $C \subset X$ is a nonzero section

$$s \in H^0(C, N^+_{C/X}) \cong H^0(\mathbb{P}^1, \mathcal{O}(1))$$

  which has a unique zero $\mathrm{Zero}(s) \in C \subset X$.
- This determines a natural holomorphic submersion $\mu : \mathbb{P}\mathcal{D} \to X$ sending s to $\mathrm{Zero}(s)$.
- We have a natural double fibration

$$\begin{array}{ccc} \mathbb{P}\mathcal{D} & \stackrel{\mu}{\longrightarrow} & X \\ \downarrow \rho & & \\ \mathcal{M} & & \end{array}$$

  giving two foliations T^ρ and T^μ of rank 1 on $\mathbb{P}\mathcal{D}$.



$\mathbb{P}D$
$\boldsymbol{\rho}$
$\boldsymbol{\mu}$
$\mathcal{M}$
$\boldsymbol{X}$

Local structure theory of Goursat distributions

- By E. Cartan's local structure theory of Goursat distributions, a Goursat distribution at a general point is isomorphic to the natural contact distribution on the space of jets of functions with one independent and one dependent variables.
- This implies that on a neighborhood $U \subset \mathbb{P}\mathcal{D}$ of a general point of $\mathbb{P}\mathcal{D}$, the data T^{ρ} and T^{μ} correspond to an ODE of order $n = \dim \mathcal{M}$ of the form ("holonomic ODE")

$$y^{(n)} = F(t, y, y^{(1)}, \ldots, y^{(n-1)}),$$

where F is a local holomorphic function in $n + 1$ variables.

Cartan connection for holonomic ODE's

Theorem (Doubrov-Komrakov-Morimoto 1999)

We can canonically associate a Cartan connection to a holonomic ODE.

- In our setting, this says that over a neighborhood $U \subset \mathbb{P}\mathcal{D}$ of a general point of $\mathbb{P}\mathcal{D}$, there exists a natural principal bundle $\mathcal{P}$ with a natural affine connection ∇ on $\mathcal{P}$.
- The naturalness implies that a formal equivalence of formal neighborhoods of rational curves induces a formal equivalence of such affine connections.

Convergence of formal equivalence of affine connections

The proof of Main Theorem is reduced to the question whether a **formal equivalence of affine connections is convergent.**

Theorem (Kobayashi-Nomizu 1963)

Let ∇ (resp. $\widetilde{\nabla}$) be an affine connection on a complex manifold Y (resp. $\widetilde{Y}$). Let

$$\psi : (y/Y)_\infty \cong (\widetilde{y}/\widetilde{Y})_\infty$$

be a formal isomorphism at points $y \in Y$ and $\widetilde{y} \in \widetilde{Y}$ such that

$$\psi_* \nabla = \widetilde{\nabla}.$$

Then ψ is convergent.

Thank you very much !!

Najlepša hvala !!