Embeddability of real and positive operators

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Motivation – credit rating

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Example (R. B. Israel, J. S. Rosenthal, J. Z. Wei 2001)

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Estimated transition matrix for credit ratings:

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$$P = \begin{pmatrix} 0.8910 & 0.0963 & 0.0078 & 0.0019 & 0.0030 & 0.0000 & 0.0000 & 0.0000 \\ 0.0086 & 0.9010 & 0.0747 & 0.0099 & 0.0029 & 0.0029 & 0.0000 & 0.0000 \\ 0.0009 & 0.0291 & 0.8894 & 0.0649 & 0.0101 & 0.0045 & 0.0000 & 0.0009 \\ 0.0006 & 0.0043 & 0.0656 & 0.8427 & 0.0644 & 0.0160 & 0.0018 & 0.0045 \\ 0.0004 & 0.0022 & 0.0079 & 0.0719 & 0.7764 & 0.1043 & 0.0127 & 0.0241 \\ 0.0000 & 0.0019 & 0.0031 & 0.0066 & 0.0517 & 0.8246 & 0.0435 & 0.0685 \\ 0.0000 & 0.0000 & 0.0116 & 0.0116 & 0.0203 & 0.0754 & 0.6493 & 0.2319 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}.$$

Here the eight columns and rows represent, in order, the credit ratings AAA, AA, A, BBB, BB, B, CCC, and Default.

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Assumption: A continuous Markov process is underlying.

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We are looking for a Markov semigroup $(P(t))_{t\geq 0}$ such that $P(1) = P$.

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Non-embeddable Markov matrix

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Suppose P is embeddable into the Markov semigroup $(P(t))_{t \geq 0}$.

Let $P\left(\frac{1}{2}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \geq 0$.

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$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = P = P(1) = P\left(\tfrac{1}{2}\right) P\left(\tfrac{1}{2}\right)$$

$$= \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix}.$$

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Comparing the entries yields a contradiction. □

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2. Uniqueness: If so, how many such semigroups are there?

Embedding in general not unique

Example (J. M. O. Speakman 1967)

Z. Wahrscheinlichkeitstheorie verw. Geb. 7, 224 (1967)

Two Markov Chains with a Common Skeleton

J. M. O. Speakman *

Received November 9, 1966

We give an example of two three-state Markov chains which coincide for some but not all $t > 0$. The first has Q-matrix (matrix of transition-probability derivatives $q_{ij} = p'_{ij}(0+)$)

$$Q_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \text{ and the second } Q_2 = \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}.$$

The characteristic equation of Q_1 is $(\lambda + 1)^3 = 1$ so that the transition functions are of the form $a + b e^{-3t/2} \cos(\sqrt{3}t/2 + \alpha)$ for some a, b and α. From considerations of symmetry we see that the asymptotic value of each of the functions is $1/3$. This and the values of the functions and of their first derivatives at 0 determine the functions completely and we have

$$p_{11}(t) = p_{22}(t) = p_{33}(t) = 1/3 + 2/3 \cdot e^{-3t/2} \cos \sqrt{3}t/2\,,$$
$$p_{12}(t) = p_{23}(t) = p_{31}(t) = 1/3 + 2/3 \cdot e^{-3t/2} \cos(\sqrt{3}t/2 - 2\pi/3) \quad \text{and}$$
$$p_{13}(t) = p_{21}(t) = p_{32}(t) = 1/3 + 2/3 \cdot e^{-3t/2} \cos(\sqrt{3}t/2 + 2\pi/3)$$

By a similar argument it can be shown that for the second chain

$$p_{11}(t) = p_{22}(t) = p_{33}(t) = 1/3 + 2/3 \cdot e^{-3t/2} \text{ and}$$

$p_{ij}(t) = 1/3 - 1/3 \cdot e^{-3t/2}$ whenever $i \neq j$. The two sets of functions coincide when $t = 4k\pi/\sqrt{3}$ where k is any integer.

Statistical Laboratory
University of Cambridge
England

* This work was done during the tenure of a Science Research Council Studentship and of a Research Studentship from Girton College.

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2. Uniqueness: If so, how many such processes are there?
3. If not embeddable, what is the "nearest" Markov process?

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Recall

A C_0-semigroup is a family of bounded linear operators $(T(t))_{t\geq 0}$ on X such that

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Now 3) and 4), where $X \in \{\mathbb{C}^n; c_0; \ell^p, 1 \leq p < \infty\}$

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For each compact $K \subseteq \mathbb{C}, K \neq \emptyset$, with $K = \overline{K}$ there exists a real-embeddable operator $T_K \in \mathcal{L}(\ell^2)$ such that $\sigma(T_K) = K$.

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Idea to show that diagonal entries are > 0

Let $T(t) = (a_{ij}(t))$. If $a_{jj}(t) = 0$ for some $t > 0$, then

$$0 = a_{jj}(t) \overset{T(t)=T\left(\frac{t}{2}\right)T\left(\frac{t}{2}\right)}{=} \sum_{k\in\mathbb{N}} a_{jk}\left(\tfrac{t}{2}\right) a_{kj}\left(\tfrac{t}{2}\right) \geq a_{jj}\left(\tfrac{t}{2}\right) a_{jj}\left(\tfrac{t}{2}\right) \geq 0.$$

By induction $a_{jj}\left(\frac{t}{2^n}\right) = 0$ for all $n \in \mathbb{N}$. We obtain the contradiction

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Necessary conditions for embeddability into positive s.g.

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However, the diagonal entries in its matrix representation on $\ell^2 \cong_{\text{pos.}} \ell^2(\mathbb{Q})$ are all 0.

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References

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