Eventual cone invariance revisited

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(joint work with Jochen Glück)

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Uniform, individual and weak eventual nonnegativity

Let X be a finite-dimensional real vector space ordered by a closed cone X_+ with non-empty interior.

Definition

A matrix semigroup $(e^{tA})_{t\geq 0}$ in X is called

(i) *uniformly eventually nonnegative* if there exists a time $t_0 \geq 0$ such that for all $t \geq t_0$ we have $e^{tA}X_+ \subseteq X_+$,

(ii) *individually eventually nonnegative* if for each $x \in X_+$ there exists $t_0 \geq 0$ such that for all $t \geq t_0$ we have $e^{tA}x \in X_+$, and

(iii) *weakly eventually nonnegative* if for each $x \in X_+$ and $x' \in X'_+$ there exists $t_0 \geq 0$ such that for all $t \geq t_0$ we have $\langle x', e^{tA}x\rangle \geq 0$.

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Since X is finite-dimensional:

$$\text{uniformly} \quad \overset{?}{\Longleftrightarrow} \quad \text{individually} \quad \overset{?}{\Longleftrightarrow} \quad \text{weakly}$$

Special Cases

$$\text{uniformly} \quad \overset{?}{\Longleftarrow} \quad \text{individually} \quad \overset{?}{\Longleftarrow} \quad \text{weakly}$$

(i) True for finite-dimensional real spaces X with polyhedral cone X_+.

(ii) A strongly continuous semigroup on a Banach lattice that is individually but not uniformly eventually nonnegative was constructed in [DGK16].
$\rightsquigarrow$ Example does not work in finite dimensions.

(iii) In finite-dimensions the equivalences can be wrong [GH23].

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Uniform vs individual eventual nonnegativity

Example (J. Glück, J.H., [GH23])

Let $X = \mathbb{R}^4$ and

$$\mathbb{R}^4_+ := \{x \mid (x_1 - x_2)^2 + x_3^2 + x_4^2 \leq x_1^2 \text{ and } x_1 \geq 0\}.$$

The semigroup

$$e^{tA} = \begin{pmatrix} 1 & t & & \\ 0 & 1 & & \\ & & \cos(2\pi t) & 2\sin(2\pi t) \\ & & -\frac{1}{2}\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

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For fixed $n \in \mathbb{N}$ consider

$$e^{(n-\frac{3}{4})A} x_n := e^{(n-\frac{3}{4})A} \begin{pmatrix} n \\ 1 \\ 0 \\ \sqrt{2n-1} \end{pmatrix} = \begin{pmatrix} 2n - \frac{3}{4} \\ 1 \\ 2\sqrt{2n-1} \\ 0 \end{pmatrix} \notin \mathbb{R}^4_+$$

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Since $x_2(t) = 1$ for all $t \geq 0$, we plot the trajectory in the affine 3D-hyperplane $\{x \in \mathbb{R}^4 \mid x_2 = 1\}$.

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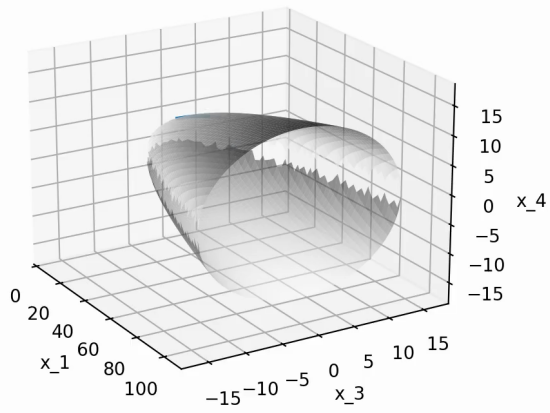
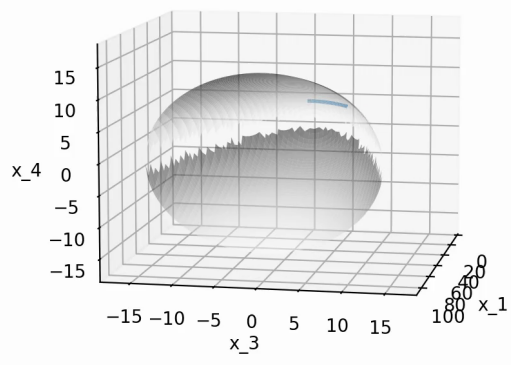
For $n = 10$ and the trajectory on the time interval $[0, 100]$:





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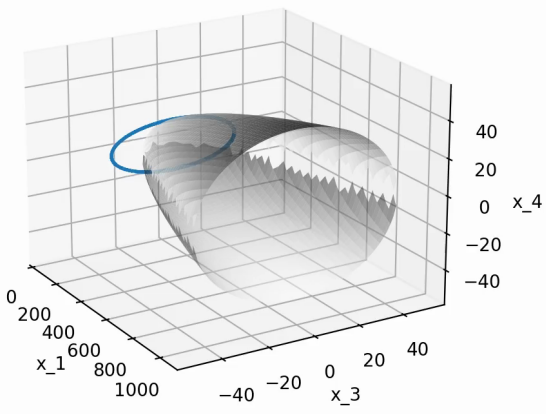
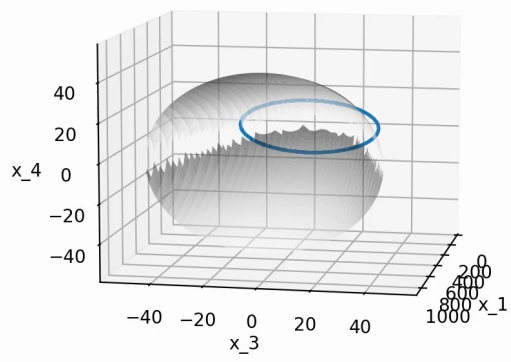
For $n = 100$ and the trajectory on the time interval $[0, 1000]$ (sped up 10 times):





Further Results

Example can be modified to yield:

$$(e^{tB})_{t\geq 0} \text{ individually} \quad \not\Longleftarrow \quad (e^{tB})_{t\geq 0} \text{ weakly}$$

and

$$(e^{tC})_{t\geq 0} \text{ individually} \quad \not\Longrightarrow \quad (e^{tC'})_{t\geq 0} \text{ individually}$$

but

$$(e^{tC})_{t\geq 0} \text{ uniformly (weakly)} \quad \Longleftrightarrow \quad (e^{tC'})_{t\geq 0} \text{ uniformly (weakly)}$$

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Bibliography

[DGK16] Daniel Daners, Jochen Glück, and James B. Kennedy. "Eventually positive semigroups of linear operators". In: *J. Math. Anal. Appl.* 433.2 (2016), pp. 1561–1593.

[GH23] Jochen Glück and Julian Hölz. "Eventual cone invariance revisited". In: *Linear Algebra and its Applications* 675 (2023), pp. 274–293.