Elliptic and parabolic operators with unbounded polynomial coefficients

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Motivation

Consider the Stochastic Differential Equation

$$\begin{cases} dX(t) = F(X(t))dt + \sigma(X(t))dW(t) \\ X(0) = x\ , \end{cases} \tag{1}$$

- Probabilistic model of the physical process of diffusion
- Model in mathematical finance
- Model in biology

$u(t,x) := \mathbb{E}(\varphi(X(t))$ satisfies Kolomogorov equation

$$\begin{cases} \partial_t u = \frac{1}{2} Tr(\sigma^T D^2 \sigma)u + F \cdot \nabla u := Lu \\ u(0) = \varphi. \end{cases} \tag{2}$$

$F = Mx$ where σ, M real matrix gives the Ornstein-Uhlembeck operator

Second order operators with polynomial coefficients

- **Unbounded Diffusion**: $A = (1+|x|^\alpha)\Delta$

  [*S. Fornaro, L. Lorenzi '07*]: $0 < \alpha \le 2$.

  [*G. Metafune, C. Spina,'10*] $\alpha > 2$, $p > \frac{N}{N-2}$

- **Unbounded Diffusion & Drift**: $A = (1+|x|^\alpha)\Delta + b|x|^{\alpha-2}x \cdot \nabla$

  [S. Fornaro, L. Lorenzi '07]: $1 < \alpha \le 2$.

  [*G. Metafune, C. Spina, C. T.,'14*] $\alpha > 2, b > 2-N \to p > \frac{N}{N-2+b}$

- **Schrödinger-Type Operator:** $A = (1+|x|^\alpha)\Delta - c|x|^\beta$

  [L. Lorenzi, A. Rhandi,'15] $0 \le \alpha \le 2, \beta \ge 0$

  [A. Canale, A. Rhandi, C.T.,'16] $\alpha > 2, \beta > \alpha - 2, 1 < p < \infty$

Second order operators with polynomial coefficients

- **Complete (degenerate)** : $A = |x|^\alpha \Delta + b|x|^{\alpha-2} x \cdot \nabla_x - c|x|^{\alpha-2}$

  [G. Metafune, N. Okazawa, M. Sobajima, C. Spina,'16]
  $N/p \in (s_1 + \min\{0, 2-\alpha\}, s_2 + \max\{0, 2-\alpha\}),\ c + s(N-2+b-s) = 0$

- **Complete** $\alpha > 2$: $A = (1+|x|^\alpha)\Delta + b|x|^{\alpha-2} x \cdot \nabla - c|x|^\beta$

  [S. Boutiah, F. Gregorio, A. Rhandi, C. T.,'18] $\alpha > 2, \beta > \alpha - 2, p > 1$

- **Complete** $\alpha < 2$: $A = (1+|x|^\alpha)\Delta + b|x|^{\alpha-2} x \cdot \nabla - c|x|^{\alpha-2} - |x|^\beta$

  [S. Boutiah, L. Caso, F. Gregorio, C. T.,'21] $\alpha \in [0, 2), \beta > 0, p > 1$

Preliminary considerations in $C_b(\mathbb{R}^N)$

We endow A with its maximal domain in C_b

$$D_{max}(A) = \{u \in C_b \cap W^{2,p}_{loc} \quad \text{for all} \quad p < \infty : \; Au \in C_b\}.$$

$$\begin{cases} u_t(t,x) = Au(t,x) & x \in \mathbb{R}^N, \; t > 0, \\ u(0,x) = f(x) & x \in \mathbb{R}^N \,, \end{cases} \tag{3}$$

$\exists \; (T_{min}(t))_{t \geq 0}$ in C_b, generated by $A_{min} = (A, \hat{D})$, where $\hat{D} \subset D_{max}$.

$$\begin{cases} D_t u_\rho(t,x) = Au_\rho(t,x) & x \in B(\rho), \;\; t > 0, \\ u_\rho(t,x) = 0 & x \in \partial B(\rho), \;\; t > 0 \\ u_\rho(0,x) = f(x) & x \in B(\rho). \end{cases} \tag{4}$$

Schauder interior estimates + compactness argument $\Longrightarrow u_{\rho_n} \to u$

The operator

For $b, c \in \mathbb{R}$, $0 \leq \alpha \leq 2$, $\beta > \alpha - 2$ and $u \in \mathcal{D}_0 = C_c^\infty(\mathbb{R}^N \setminus \{0\})$

$$\mathcal{A}u := (1 + |x|^\alpha)\Delta u + b|x|^{\alpha-2}x \cdot \nabla u - c|x|^{\alpha-2}u - |x|^\beta u$$

Aims.

- $(A, D_p(A))$ an extension $(\mathcal{A}, \mathcal{D}_0)$ generates analytic C_0-semigroup on $L^p(\mathbb{R}^n)$, $1 < p < \infty$
- conditions on the coefficients under which this extension is precisely the closure of $(\mathcal{A}, \mathcal{D}_0)$ and give characterization of $D_p(A)$

Tools

- form method $\Longrightarrow$ generation of a semigroup $T(t)$ on $L^2(\mathbb{R}^N)$
- sub-Markovian properties i.e. $T(t) \geq 0$ and $\|T(t)\|_\infty \leq 1 \Longrightarrow$ extrapolation to $L^p(\mathbb{R}^N)$
- Okazawa perturbation Theorem $\Longrightarrow$ domain of the extension Au is perturbation of $(1 + |x|^\alpha)\Delta u - |x|^\beta u$

Form methods for semigroups

H Hilbert space, $\mathfrak{a} : D(\mathfrak{a}) \times D(\mathfrak{a}) \to \mathbb{C}$ sesquilinear form

- densely defined, i.e. $D(\mathfrak{a})$ is dense in H
- accretive i.e. $Re\mathfrak{a}(u,u) \geq 0$
- continuos w.r. $\|u\|_{\mathfrak{a}} = \sqrt{Re\mathfrak{a}(u,u) + \|u\|^2}$
- closed i.e. $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is complete

define $A : D(A) \to H$ such that

$$\mathfrak{a}(u,v) =< Au, v > \text{ for all } v \in D(\mathfrak{a})$$

where $D(A) = \{u \in D(\mathfrak{a}),\ \exists f \in H\, s.t.\, \mathfrak{a}(u,v) =< f, v > \forall v \in D(\mathfrak{a})\}$

Theorem (Generation Theorem via forms)

$-A$ generate an analytic contraction semigroup e^{-tA}

Construction of the Form

Since for $u, v \in \mathcal{D}_0 = C_c^\infty(\mathbb{R}^N \setminus \{0\})$ the following holds

$$-\int_{\mathbb{R}^N} Au\overline{v}\,dx = \int_{\mathbb{R}^N} \Big((1+|x|^\alpha)\Delta u + b|x|^{\alpha-2}x\cdot\nabla u - c|x|^{\alpha-2}u - |x|^\beta u\Big)\,\overline{v}d$$

$$= \int_{\mathbb{R}^N} \Big((1+|x|^\alpha)\nabla u\cdot\nabla\overline{v} + (\alpha-b)|x|^{\alpha-2}x\cdot\nabla u\overline{v}$$

$$+c|x|^{\alpha-2}u\overline{v} + |x|^\beta u\overline{v}\Big)\,dx.$$

we define the folloing bilinear form

$$\mathfrak{a}(u,v) = \int_{\mathbb{R}^N} \Big((1+|x|^\alpha)\nabla u\cdot\nabla\overline{v} + (\alpha-b)|x|^{\alpha-2}x\cdot\nabla u\,\overline{v}$$

$$+c|x|^{\alpha-2}u\overline{v} + |x|^\beta u\overline{v} + \lambda u\overline{v}\Big)\,dx,$$

$$D(\mathfrak{a}) = \left\{u\in H^1(\mathbb{R}^N)\,:\,(1+|x|^\alpha)^{\frac{1}{2}}\nabla u, \left(|x|^{\alpha-2}\right)^{\frac{1}{2}}u, \left(|x|^\beta\right)^{\frac{1}{2}}u \in L^2\left(\mathbb{R}^N\right)\right\}$$

where λ is a suitable positive constant that will be chosen later.

Let us compute $\mathfrak{a}(u,u)$

$$\begin{aligned}
&\operatorname{Re}\mathfrak{a}(u,u)\\
&\quad=\int_{\mathbb{R}^N}\Big((1+|x|^\alpha)|\nabla u|^2\\
&\qquad+\left[\left(c-\frac{\alpha-b}{2}(\alpha-2+N)\right)|x|^{\alpha-2}+|x|^\beta+\lambda\right]|u|^2\Big)\,dx.
\end{aligned}$$

Hardy inequality $c_0\int_{\mathbb{R}^N}\frac{u^2}{x^2}dx\le\int_{\mathbb{R}^N}|\nabla u|^2dx$ gives

$$\begin{aligned}
\operatorname{Re}\mathfrak{a}(u,u)\ge&\int_{\mathbb{R}^N}|x|^\alpha|\nabla u|^2\,dx\;+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx\\
&+\int_{\mathbb{R}^N}\left[\frac{c_0}{2|x|^2}+\left(c-1-\frac{\alpha-b}{2}(\alpha-2+N)\right)|x|^{\alpha-2}+\lambda\right]|u|^2\,dx\\
&+\int_{\mathbb{R}^N}|x|^{\alpha-2}|u|^2\,dx+\int_{\mathbb{R}^N}|x|^\beta|u|^2\,dx\ge 0
\end{aligned}$$

For a suitable $\lambda>0$

For the form so defined we have the following

Theorem

The form $\mathfrak{a}$ is densely defined, accretive, continuous and closed. Therefore, it is associated to a closed operator $(-A_\lambda, D(A_\lambda))$ on $L^2(\mathbb{R}^N)$

$$D(A_\lambda) := \{u \in D(\mathfrak{a}) \,:\, \exists\, v \in L^2(\mathbb{R}^N)\, s.t.\ \mathfrak{a}(u,h) = \langle v, h\rangle,\ \forall\, h \in D(\mathfrak{a})\}$$
$$-A_\lambda u := v.$$

Moreover, $(A_\lambda, D(A_\lambda))$ is the generator of a strongly continuous analytic contraction semigroup e^{tA_λ} on $L^2(\mathbb{R}^N)$.

Now we need to prove that $(A_\lambda, D(A_\lambda))$ is an extension of $(\mathcal{A} - \lambda, \mathcal{D}_0)$.

$$-\int_{\mathbb{R}^N} (\mathcal{A}u - \lambda u)\overline{h}\, dx = \mathfrak{a}(u,h) \text{ for all } u, h \in \mathcal{D}_0$$

If $\mathcal{D}_0$ is a core for $\mathfrak{a}$ then $A_\lambda \equiv A - \lambda$ on $\mathcal{D}_0$

Proposition

$\mathcal{D}_0$ is a core for $\mathfrak{a}$.

PROOF. Take $u \in D(\mathfrak{a})$ and consider $u_n = u\varphi_n$, where $\varphi_n \in \mathcal{D}_0$

$$\begin{cases} \varphi_n = 0 \text{ in } B(\frac{1}{n}) \cup B^c(2n), \\ \varphi_n = 1 \text{ in } B(n) \setminus B(\frac{2}{n}), \\ 0 \le \varphi_n \le 1, \\ |\nabla \varphi_n(x)| \le C\frac{1}{|x|}. \end{cases}$$

$$(1+|x|^\alpha)^{\frac{1}{2}}|\nabla u_n - \nabla u| \le (1+|x|^\alpha)^{\frac{1}{2}}(1-\varphi_n)|\nabla u| + (1+|x|^\alpha)^{\frac{1}{2}}|u||\nabla \varphi_n|$$
$$\le (1+|x|^\alpha)^{\frac{1}{2}}(1-\varphi_n)|\nabla u| + \left(\frac{|u|}{|x|} + |x|^{\frac{\alpha}{2}-1}|u|\right)\chi_{k_n}$$

where $k_n = B\left(\frac{2}{n}\right) \setminus B\left(\frac{1}{n}\right) \cup B(2n) \setminus B(n)$.
Then $u_n \in H_0^1(\mathbb{R}^N \setminus \{0\})$ and $\|u_n - u\|_{\mathfrak{a}} \to 0$ by dominated convergence.

Extrapolation to $L^p(\mathbb{R}^N)$

Extend the family $(e^{tA_\lambda})_{t\geq 0}$ of bounded operators $L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ to a family of bounded operators $L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$.

Case positive potential $c|x|^{\alpha-2}$, $c \geq 0$.

- $(e^{tA_\lambda})_{t\geq 0}$ is sub-Markovian i.e. $e^{tA_\lambda} \geq 0$ and $\|e^{tA_\lambda}\|_\infty \leq 1$
- by Riesz-Thorin interpolation theorem e^{tA_λ} can be extended to an operator $S_p(t)$ on L^p for every $2 \leq p \leq \infty$
- $S_p(t)$ defines a consistent family of C_0-semigroup of contractions in $L^p(\mathbb{R}^N)$ for $2 \leq p < \infty$
- by duality e^{tA_λ} can be extended to an operator on $L^p(\mathbb{R}^N)$ for every $1 < p \leq 2$

Tools

$u \in D(\mathfrak{a}) \cap L^2(\mathbb{R}^N) \implies u^+ \in D(\mathfrak{a})$ and $\mathfrak{a}(u^+, u^-) \leq 0$ give positivity

$u \in D(\mathfrak{a}) \cap L^2(\mathbb{R}^N)^+ \implies 1 \wedge u \in D(\mathfrak{a}), \mathfrak{a}(1 \wedge u, (u-1)^+) \geq 0$ give $L^\infty-$contractivity

Case negative potential $c|x|^{\alpha-2}$, $c \le 0$. Let $A_0 = (1+|x|^\alpha)\Delta u + b|x|^{\alpha-2}x\cdot\nabla - |x|^\beta$, consider

$$A_n - \lambda = A_0 - \lambda - W_n$$

where $W_n = max\{-n, c|x|^{\alpha-2}\}$
$A_0 - \lambda$ generates a semigroup $e^{t(A_0-\lambda)}$ in L^p (by the previous point).
The sum $A_0 - \lambda - W_n$ generate a C_0-semigroup

$$0 \le e^{t(A_0-\lambda-W_n)} \le e^{t(A_0-\lambda-W_{n+1})} \to S(t) = e^{tA_\lambda}$$

Theorem

Let $N \ge 3, \alpha \in (0,2), \beta > 0, b, c \in \mathbb{R}$. There exists $(A, D_p(A))$, an extension of $(\mathcal{A}, \mathcal{D}_0)$, that generates an analytic C_0-semigroup in $L^p(\mathbb{R}^N)$ for any $1 < p < \infty$.

$$e^{tA} := e^{\lambda t} e^{tA_\lambda}$$

Remark

if $p(\alpha-2) > -N$ that is $\mathcal{A}u \in L^p$ for $u \in \mathcal{D} = C_c^\infty(\mathbb{R}^N)$, then $(A, D_p(A))$

Domain Characterization

Condition s.t. $(A, D_p(A))$ coincides with the closure of $(\mathcal{A}, \mathcal{D}_0)$.

$$-A = -A_0 + cW = -(1+|x|^\alpha)\Delta + |x|^\beta + c|x|^{\alpha-2}$$

- *L. Lorenzi, A. Rhandi,'15* generation results for $(A_0, D(A_0))$
- Okazawa perturbation theorem $-A$ m-accretive on $D(A_0)$
- Let $\phi = (1+|x|^\alpha)^{b/\alpha}$, seting $u = \frac{v}{\sqrt{\phi}}$, give the drift term
  $(1+|x|^\alpha)\Delta u + b|x|^{\alpha-2}x \cdot \nabla u - c|x|^{\alpha-2}u - |x|^\beta u$

Theorem

Let $0 \le \alpha < 2$ and either $1 < p < \frac{N-\alpha}{2-\alpha}$ or else $p = \frac{N-\alpha}{2-\alpha}$ and

$$\left(\frac{N}{p} - 2 + \alpha\right)\left(\frac{N}{p'} - \alpha + b\right) + c > 0. \tag{5}$$

Then the closure of $(A, \mathcal{D})$ coincide with $(A, D_p(A))$ and generates an analytic C_0-semigroup.

Theorem (Okazawa)

Let A and B be linear $m-$accretive operators in L^p. Let D be a core for A and let $B_\varepsilon = \frac{1}{\varepsilon}B(\frac{1}{\varepsilon}+B)^{-1}$ be the Yosida approximation of B.

(i) *there are constants $a_1, a_2 \geq 0$ and $k_1 > 0$ s. t. for all $u \in D$*

$$\mathrm{Re}\langle Au, F(B_\varepsilon u)\rangle \geq k_1\|B_\varepsilon u\|_p^2 - a_2\|u\|_p^2 - a_1\|B_\varepsilon u\|_p\|u\|_p$$

Then B is $A-$bounded with $A-$bound k_1^{-1}:

$$\|Bu\|_p \leq k_1^{-1}\|Au\|_p + k_0\|u\|, \quad u \in D(A) \subset D(B).$$

Assume further that

(ii) $\mathrm{Re}\langle u, F(B_\varepsilon u)\rangle \geq 0$, *for all $u \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$;*

(iii) *there is $k_2 > 0$ such that $A - k_2 B$ is accretive.*

Set $k = \min\{k_1, k_2\}$. If $t > -k$ then $A + tB$ with domain $D(A + tB) = D(A)$ is m-accretive and any core of A is also a core for $A + tB$. Furthermore, $A - kB$ is essentially m-accretive on $D(A)$.

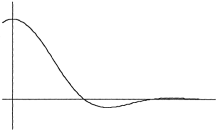
Higher order operators

- models of elasticity, non-linear elasticity
- condensation in graphene
- free boundary problems

$$u_t(t,x) = -\Delta^2 u(t,x)$$

Interesting mathematical features:

- no maximum principles;
- no positivity preserving properties:
  $p(t,x) = t^{-\frac{1}{4}} p(1, t^{-\frac{1}{4}} x)$ with

$p(1,\cdot) =$ 

(Davies 1995)

- no classical Markov semigroup theory (Sobolev inequalities, L^∞-contractivity).

Eventual positivity of $e^{-t\Delta^2}$

Eventual positivity : Positivity for large enough time
Local eventual positivity : Eventual positivity on compact set

Gazzola-Grunau, Discr. Cont. Dyn. Syst., 2008

Proved that $e^{-t\Delta^2}$ is
Individually locally eventually positive: Let $0 \le u_0 \in C_c(\mathbb{R}^N)$

- for any compact $K \subset \mathbb{R}^N$, $\exists T_K > 0$ that depends on u_0 s.t. $e^{-t\Delta^2} u_0(x) > 0$ for all $t \ge T_K$, $x \in K$;
- $\exists \tau > 0$ that depends on u_0 such that for any $t > \tau$, $\exists x_t \in \mathbb{R}^N$ s.t. $e^{-t\Delta^2} u_0(x_t) < 0$.

General abstract theory

Daners, D., Glück, J., Kennedy, J.B., 2016

Bi-Kolmogorov operator

D. Addona, F. Gregorio, A. Rhandi, C. T., NoDEA, 2022

Consider the Kolmogorov operator

$$L := \Delta + \frac{\nabla \mu}{\mu} \cdot \nabla$$

and $A = L^2$ in the $L^2(\mathbb{R}^N, d\mu) = L^2_\mu$ setting

- Weighted Rellich's inequality
  $(C_0 - 1)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} d\mu \leq \int_{\mathbb{R}^N} |Lu|^2 d\mu + C\|u\|^2_{H^1_\mu}, \qquad u \in C_c^\infty(\mathbb{R}^N)$
- Generation results in L^2_μ for $-A$ via form method
  $a_L(u, v) := \int_{\mathbb{R}^N} Lu\overline{Lv}\, d\mu, \quad u, v \in D(L)$
- $d\mu$ is the unique invariant measure
  $\int_{\mathbb{R}^N} e^{-tA} f\, d\mu = \int_{\mathbb{R}^N} f\, d\mu, \quad f \in L^2_\mu$
- Domain characterization $D(L) = H^2(\mathbb{R}^N, d\mu)$, $D(A) = H^4(\mathbb{R}^N, d\mu)$
- Asymptotic properties and eventually positivity of e^{-tA}
- Heat kernel of bi-Ornstein-Ulhenbeck semigroup
  $\mu(x) = (2\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2}}$ and $Lu = \Delta u - x \cdot \nabla u$

Analysis of $(e^{-tA})_{t\geq 0}$ and asymptotic behavior

Proposition

0 is an eigenvalue of A, and the corresponding eigenspace consists of constant functions.

Proposition

μ is ergodic with respect to the semigroup $(e^{-tA})_{t\geq 0}$

$$L^2 - \lim_{t\to+\infty} \frac{1}{t}\int_0^t e^{-sA} f ds = \int_{\mathbb{R}^N} f d\mu, \quad f \in L^2_\mu(\mathbb{R}^N).$$

Proposition

For $f \in L^2_\mu$ one has

$$L^2_\mu - \lim_{t\to\infty} e^{-tA} f = \int_{\mathbb{R}^N} f\, d\mu \,(= P_\infty f)$$

Eventual positivity and Asymptotic behaviour of e^{-tA}

- Spectral properties $\Longrightarrow$ *Individual asymptotic positivity*

$$f \in L^2_\mu(\mathbb{R}^N)_+ \Rightarrow \lim_{t\to+\infty} \mathrm{dist}(e^{-tA}f, L^2_\mu(\mathbb{R}^N)_+) = 0,$$

- Asymptotic behaviour $\Longrightarrow$ *asymptotically irreducible*

$$f \in L^2_\mu(\mathbb{R}^N)_> \Rightarrow \lim_{t\to+\infty} \mathrm{dist}(e^{-tA}f, L^2_\mu(\mathbb{R}^N)_>) = 0$$

$$L^2_\mu(\mathbb{R}^N)_> := \{f \in L^2_\mu(\mathbb{R}^N)_+ : \exists A, |A| > 0,\ f(x) > 0\ x \in A\}$$

Eventual positivity and Asymptotic behaviour of e^{-tA}

- *Locally uniformly eventually positivity.*

$K \subset \mathbb{R}^N$ compact, $\exists t_0 > 0$ s.t. $\forall f \in L^2_\mu(K)_> \ \exists c > 0$

$$e^{-tA}(\chi_K f)(x) \geq c, \quad t \geq t_0, \text{ a.e. in } x \in K.$$

By recent results by Arora, 2022

Proposition

Assume that

i) there exists $n \in \mathbb{N}$ such that $D(A^n) \subset L^\infty_{\mathrm{loc}}(\mathbb{R})$

ii) 0 is a simple pole for $\sigma(-A)$.

then the semigroup is locally uniformly eventually positive.

Sobolev embedding, more regularity assumption on $\mu \Longrightarrow i)$
A has compact resolvent, 0 eigenvalue, $1-$dim eigenspace $\Longrightarrow ii)$

The bi-Ornstein-Uhlenbeck operator

$\mu(x) = (2\pi)^{-N/2} e^{-|x|^2/2} \Longrightarrow$

$$\begin{aligned} Lu &= \Delta u - x \cdot \nabla u \\ Au &= \Delta^2 u - 2x \cdot \nabla(\Delta u) + Tr(x \otimes x D^2 u) \\ &\quad -2\Delta u + x \cdot \nabla u \end{aligned}$$

[Lunardi 1997] $\Longrightarrow\ D(L) = H^2_\mu(\mathbb{R}^N)$.
Characterization of domain $\Longrightarrow\ D(A) = H^4_\mu(\mathbb{R}^N)$.

Remark The bi-Ornstein-Uhlenbeck operator require $N \geq 5$

Heat kernel of bi-Ornstein-Uhlenbeck

$$
\begin{aligned}
e^{-tA}f(x) &= \int_{\mathbb{R}^N} k(t,x,y)f(y)\,dy \\
= \quad & \sqrt{2}(8\pi)^{-\frac{N+1}{2}} \int_0^\infty e^{-\frac{s^2}{4}}(\sin(s\sqrt{t}))^{-N/2} e^{-\frac{|e^{-is\sqrt{t}}x-y|^2}{8}} \\
& \cos\left(\frac{N}{2}(s\sqrt{t}-\frac{\pi}{2}) + \frac{|e^{-is\sqrt{t}}x-y|^2}{8\tan(s\sqrt{t})}\right) ds
\end{aligned}
$$

If L generate an analytic semigroup $T(t)$ of angle ϑ then $e^{\pm i\vartheta}L$ generate the C_0-semigroup $T(e^{\pm i\vartheta}s)$ called "Boundary" semigroup.
If the angle is $\frac{\pi}{2}$ then $\pm iL$ generate the semiroups $T(\pm is)$ and then iL generates a group $T(is)$ for $s \in \mathbb{R}$.
Then $(iL)^2 = -A$ generate a semigroup, and the kernel is given by

$$
e^{-tA} = \int_{\mathbb{R}} \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|s|^2}{4t}} T(is)ds
$$

Many thanks