Quasi-modular spaces and copies of l_∞

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Quasi-modular spaces - filling the existing gap

- (1) The notion of a modular leads to the F-norm ([1]).
  (2) The notion of a convex modular leads to the norm ([1]).

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- Which functional leads (in a natural way) in the context of modular spaces to the quasi-norm?

The outline of the talk

1. Introduction.
2. Quasi-modular spaces.
3. Quasi-normed Calderón–Lozanovskiĭ spaces E_φ.
4. Isomorphic and isometric copies of l_∞ in E_φ.

The talk is based on the paper :

[2] P. Foralewski, H. Hudzik and P. Kolwicz, *Quasi-modular spaces with application to quasi-normed Calderón–Lozanovskiĭ spaces*, submitted.

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Introduction.

- Given a real vector space X the functional $x \mapsto \|x\|$ is called a *quasi-norm* if the following conditions are satisfied:
  (i) $\|x\| = 0$ if and only if $x = 0$;
  (ii) $\|ax\| = |a|\|x\|$ for any $x \in X$ and $a \in \mathbb{R}$.;
  (iii) there exists $C = C_X \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

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- For $0 < p \leq 1$, the functional $x \mapsto \|x\|_1$ is called a *p-norm* if it satisfies the first two conditions of the quasi-norm and the condition $\|x + y\|_1^p \leq \|x\|_1^p + \|y\|_1^p$ for any $x, y \in X$.

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- Each p-norm is a quasi-norm. By the *Aoki–Rolewicz theorem* , given a quasi-norm $\|\cdot\|$, if $0 < p \leq 1$ is such that $C = 2^{1/p-1}$, then there exists a p-norm $\|\cdot\|_1$ which is equivalent to $\|\cdot\|$.

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- The quasi-norm $\|\cdot\|$ induces a metric topology on X: in fact a metric can be defined by $d(x, y) = \|x - y\|_1^p$. We say that $X = (X, \|\cdot\|)$ is a *quasi-Banach space* if it is complete for this metric.

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- Examples: L_p, l_p for $0 < p < 1$, $L_\varphi, l_\varphi, \Lambda_w, \lambda_w$.

Building the definition of quasi-modular ...

Quasi-modular spaces

Definition. Let X be a real linear space. We say that a functional $\rho : X \to [0, \infty]$ is a quasi-modular whenever for all $x, y \in X$ the following conditions are satisfied:

(i) $\rho(0) = 0$ and the condition $\rho(\lambda x) \leq 1$ for all $\lambda > 0$ implies that $x = 0$.

(ii) $\rho(-x) = \rho(x)$.

(iii) $\rho(\lambda x)$ is non-decreasing function of λ, where $\lambda \geq 0$.

(iv) There is $M \geq 1$ such that

$$\rho(\alpha x + \beta y) \leq M [\rho(x) + \rho(y)]$$

provided $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

(v) There is a constant $p > 0$ such that for all $\varepsilon > 0$ and all $A > 0$ there exists $K = K(\varepsilon, A) \geq 1$ such that

$$\rho(ax) \leq K a^p \rho(x) + \varepsilon$$

for any $0 < a \leq 1$ whenever $\rho(x) \leq A$.

Quasi-modular spaces

- **Definition.** If ρ is a quasi-modular on X, then

$$X_\rho := \left\{ x \in X \colon \lim_{\lambda \to 0} \rho(\lambda x) = 0 \right\} \tag{1}$$

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- **Lemma.** For any quasi-modular ρ, we have

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- **Lemma.** For any quasi-modular ρ, we have

$$X_\rho = \{ x \in X \colon \rho(\lambda x) < \infty \text{ for some } \lambda > 0 \} . \tag{2}$$

- **Theorem.** Let ρ be a quasi-modular on X. Then the functional

$$\|x\|_\rho = \inf \{ \lambda > 0 : \rho\,(x/\lambda) \leq 1 \}$$

is a quasi-norm on X_ρ.

Quasi-normed Calderón–Lozanovskiĭ spaces.

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- (T, Σ, μ) is a complete σ-finite measure space and $L^0 = L^0(\mu)$ is the space of all (equivalence) classes of Σ-measurable real-valued functions defined on Ω.
- A quasi-normed lattice [quasi-Banach lattice] $E = (E, \leq, \|\cdot\|_E)$ is called a *quasi-normed ideal space [quasi-Banach ideal space (or a quasi-Köthe space)]* if it is a linear subspace of L^0 satisfying the following conditions:
  (i) if $x \in L^0$, $y \in E$ and $|x| \leq |y|$ μ-a.e., then $x \in E$ and $\|x\|_E \leq \|y\|_E$.
  (ii) There exists $x \in E$ which is strictly positive on the whole T (weak unit).

- A function $\varphi : [0, \infty) \to [0, \infty]$ is called an *Orlicz function* if φ is non-decreasing, vanishing and right continuous at 0, continuous on $(0, b_\varphi)$, where

$$b_\varphi = \sup \{u \geq 0 : \varphi(u) < \infty\}$$

and left continuous at b_φ. We will assume also that

$$\lim_{u \to \infty} \varphi(u) = \infty.$$

Let

$$a_\varphi = \sup \{u \geq 0 : \varphi(u) = 0\}.$$

Quasi-normed Calderón–Lozanovskiĭ spaces.

Recall that for any Orlicz function φ the *lower Matuszewska–Orlicz index* α_φ **for all arguments** is defined by the formula

$$\alpha_\varphi^a = \sup\{p \in \mathbb{R} : \text{there exists } K \geq 1 \text{ such that}$$
$$\varphi(au) \leq K a^p \varphi(u) \text{ for any } u \in \mathbb{R} \text{ and } 0 < a \leq 1\}. \tag{3}$$

Analogously the *lower Matuszewska–Orlicz indexes* **for large** and **for small arguments** are defined as

$$\begin{aligned}\alpha_\varphi^\infty &= \sup\{p \in \mathbb{R} : \text{there exist } K \geq 1 \text{ and } u_0 > 0 \text{ such that } \varphi(u_0) < \infty \\ &\quad \text{and } \varphi(au) \leq K a^p \varphi(u) \text{ for any } u \geq u_0 \text{ and } 0 < a \leq 1\}\end{aligned}$$

and

$$\begin{aligned}\alpha_\varphi^0 &= \sup\{p \in \mathbb{R} : \text{there exist } K \geq 1 \text{ and } u_0 > 0 \text{ such that} \\ &\quad \varphi(au) \leq K a^p \varphi(u) \text{ for any } u \leq u_0 \text{ and } 0 < a \leq 1\},\end{aligned}$$

respectively.

Quasi-normed Calderón–Lozanovskiĭ spaces.

- **Definition.** For any pair E and φ we define the index α_φ^E by the formula

$$\alpha_\varphi^E := \begin{cases} \alpha_\varphi^a, & \text{when neither } L_\infty \subset E \text{ nor } E \subset L_\infty, \\ \alpha_\varphi^\infty, & \text{when } L_\infty \subset E, \\ \alpha_\varphi^0, & \text{when } E \subset L_\infty. \end{cases} \tag{4}$$

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- **Definition.** Given a quasi-Banach ideal space E and an Orlicz function φ, we define on L^0 a functional ρ_φ^E, by

$$\rho_\varphi^E(x) := \begin{cases} \|\varphi(|x|)\|_E & \text{if } \varphi(|x|) \in E, \\ \infty & \text{otherwise.} \end{cases} \tag{5}$$

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- **Theorem.** Let E be a quasi-Banach ideal space and φ be an Orlicz function. If $\alpha_\varphi^E > 0$, then ρ_φ^E is a quasi-modular.

Quasi-normed Calderón–Lozanovskiĭ spaces.

- **Definition.** Let a quasi-Banach ideal space E and an Orlicz function φ be such that $\alpha_\varphi^E > 0$. Then the Calderón–Lozanovskiĭ space E_φ is defined by

$$E_\varphi = \{x \in L^0 : \lim_{\lambda \to 0} \rho_\varphi^E(\lambda x) = 0\}. \tag{6}$$

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- Moreover, the functional

$$\|x\|_\varphi = \inf\left\{\lambda > 0 : \rho_\varphi^E\left(x/\lambda\right) \le 1\right\}, \tag{8}$$

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- It is easy to show, that $E_\varphi = (E_\varphi, \le, \|\cdot\|_\varphi)$ is a quasi-normed ideal space.

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- If $E = L^1$ $(E = l^1)$ then E_φ is the Orlicz space L_φ (l_φ) with the Luxemburg-Nakano quasi-norm.
- If $E = \Lambda_w$ $(E = \lambda_w)$, the (quasi-normed) Lorentz space, then E_φ is the (quasi-normed) Orlicz-Lorentz space $L_{\varphi,w}$ $(l_{\varphi,w})$.

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- We say that a quasi-normed ideal space E has the *Fatou property* , if for any $x \in L^0$ and any $(x_n)_{n=1}^{\infty}$ in E_+ such that $x_n \uparrow |x|$ μ-a.e and $\sup_{n\in N} \|x_n\|_E < \infty$, we get $x \in E$ and $\lim_n \|x_n\|_E = \|x\|_E$.

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- If $E \in (FP)$ then $E_\varphi \in (FP)$, whence E_φ is complete, so $E_\varphi = (E_\varphi, \leq, \|\cdot\|_\varphi)$ is a quasi-Banach ideal space.

Quasi-normed Calderón–Lozanovskiĭ spaces.

- The symbol E_a denotes the subspace of all order continuous elements in E, that is

$$E_a = \{x \in E : 0 \leq u_n \leq |x| \text{ and } u_n \to 0 \ \mu\text{-a.e. implies } \|u_n\|_E \to 0\}.$$

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- We say that an *quasi-normed ideal space* E is order continuous ($E \in (OC)$ for short) if $E = E_a$.

- **Definition.** Recall that an Orlicz function φ satisfies the condition Δ_2 **for all** $u \in \mathbb{R}_+$ ($\varphi \in \Delta_2(\mathbb{R}_+)$ for short) if there exists a constant $K > 0$ such that the inequality

$$\varphi(2u) \leq K\varphi(u) \tag{9}$$

holds for any $u \in \mathbb{R}_+$ (then we have $a_\varphi = 0$ and $b_\varphi = \infty$). Analogously, we say that an Orlicz function φ satisfies the condition Δ_2 **for large [for small]** ($\varphi \in \Delta_2(\infty)$ [$\varphi \in \Delta_2(0)$] for short) if there exist constants $K, u_0 \in (0, \infty)$ such that $\varphi(u_0) < \infty$ [$\varphi(u_0) > 0$] and inequality (9) holds for any $u \geq u_0$ [$0 \leq u \leq u_0$], respectively.

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- For any quasi-Banach ideal space E and any Orlicz function φ we say that φ satisfies the condition Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:
(1) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L_\infty \subset E$ nor $E \subset L_\infty$,
(2) $\varphi \in \Delta_2(\infty)$ whenever $L_\infty \subset E$,
(3) $\varphi \in \Delta_2(0)$ whenever $E \subset L_\infty$.

Quasi-normed Calderón–Lozanovskiĭ spaces.

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- **Theorem.** (i) Let μ be nonatomic, $L_\infty \subset E$ and $E_a \neq \{0\}$. If $\varphi \notin \Delta_2^E$, then the space E_φ contains an order linearly isometric copy of l_∞.
  (ii) Let μ be nonatomic. Assume that neither $L_\infty \subset E$ nor $E \subset L_\infty$ and $\mathrm{supp}(E_a) = T$. If $\varphi \notin \Delta_2^E$, then the space E_φ contains an order linearly isometric copy of l_∞.

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- **Theorem.** Let μ be nonatomic. Then E_φ contains an order isomorphic copy of l_∞ if and only if E contains an order isomorphic copy of l_∞ or $\varphi \notin \Delta_2^E$.

- Denote by P the property of having the linear order isometric copy of l_∞.
  Already known: if $\varphi \notin \Delta_2^E$, then $E_\varphi \in (P)$ (under some natural assumptions on E).

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- To study these questions the following implications are crucial:
  (1) if $\rho_\varphi^E(x) = 1$ then $\|x\|_\varphi = 1$ (automatically true if φ is convex).
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- (2) $\rightarrow$ the condition Δ_{2-str}^E.

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$$\varphi(\varepsilon u) \leq \delta \varphi(u) \tag{10}$$

holds for any $u \geq 0$. We say that φ satisfies the condition Δ_ε **for large [for small]** ($\varphi \in \Delta_\varepsilon(\infty)$ [$\varphi \in \Delta_\varepsilon(0)$] for short) if for any $\varepsilon \in (0,1)$ there exist $\delta = \delta(\varepsilon) \in (0,1)$ and $u_0 = u_0(\varepsilon) > 0$ such that inequality holds for any $u \geq u_0$ [$0 \leq u \leq u_0$], respectively ([CHKK, 2022]).

Quasi-normed Calderón–Lozanovskiĭ spaces.

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- For any quasi-Banach ideal space E and any Orlicz function φ we say that φ satisfies the condition Δ_ε^E ($\varphi \in \Delta_\varepsilon^E$ for short) if:
  (1) $\varphi \in \Delta_\varepsilon(\mathbb{R}_+)$ whenever neither $L_\infty \subset E$ nor $E \subset L_\infty$,
  (2) $\varphi \in \Delta_\varepsilon(\infty)$ whenever $L_\infty \subset E$,
  (3) $\varphi \in \Delta_\varepsilon(0)$ whenever $E \subset L_\infty$.

- If φ is convex then $\varphi \in \Delta_{\varepsilon}(\mathbb{R}_+)$.

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- If φ is convex then $\varphi \in \Delta_{\varepsilon}(\mathbb{R}_+)$.
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- First

$$\varphi \in \Delta_2(\infty) \not\Rightarrow \varphi \in \Delta_\varepsilon(\infty)$$

Every monotone upper bounded Orlicz function φ satisfies the $\Delta_2(\infty)$-condition but does not satisfy the $\Delta_\varepsilon(\infty)$-condition ([1]).

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  $$\varphi \in \Delta_\varepsilon(\infty) \not\Rightarrow \varphi \in \Delta_2(\infty)$$

  Take $\varphi(u) = e^{u^2} - 1$.

Quasi-normed Calderón–Lozanovskiĭ spaces.

- If $E \subset L_\infty$, the this inclusion is continuous, whence there exists a constant $D_E > 0$ such that

$$\|x\|_{L_\infty} \leq D_E \|x\|_E \tag{11}$$

for each $x \in E$. We denote

$$a_E = \inf \{ \|\chi_A\|_E : \chi_A \in E, \mu(A) > 0 \} . \tag{12}$$

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- **Lemma.** Suppose one of the following three conditions holds:
(i) $\varphi \in \Delta_\varepsilon^E$, whenever neither $L_\infty \subset E$ nor $E \subset L_\infty$,
(ii) $\varphi \in \Delta_\varepsilon(\mathbb{R}_+)$ or $(\varphi \in \Delta_\varepsilon^E$, φ is strictly increasing on the interval (a_φ, b_φ) and there exists a constant $B > 0$ such that $|x(t)| \geq B$ for μ-a.e. $t \in T)$, whenever $L_\infty \subset E$,
(iii) $\varphi \in \Delta_\varepsilon^E$ and φ is strictly increasing on the interval $(a_\varphi, \min(\varphi^{-1}(1/a_E), b_\varphi))$, where a_E is defined by formula (12), whenever $E \subset L_\infty$.
Then, for $x \in E_\varphi$, if $\rho_\varphi^E(x) = 1$, then $\|x\|_\varphi = 1$.

- **Theorem.** Suppose one of the following three conditions holds:
(i) $\varphi \in \Delta_{\varepsilon}^{E}$, whenever neither $L_\infty \subset E$ nor $E \subset L_\infty$,
(ii) $\varphi \in \Delta_{\varepsilon}(\mathbb{R}_+)$ or $(\varphi \in \Delta_{\varepsilon}^{E}$, φ is strictly increasing on the interval (a_φ, b_φ) and E is rearrangement invariant Banach space over $T = (0,1)$ or $T = (0,\infty)$ with μ being the Lebesgue measure such that $\operatorname{supp} E_a = \operatorname{supp} E)$, whenever $L_\infty \subset E$,
(iii) $\varphi \in \Delta_{\varepsilon}^{E}$ and φ is strictly increasing on the interval $(a_\varphi, \min(\varphi^{-1}(1/a_E), b_\varphi))$, where a_E is defined by formula (12), whenever $E \subset L_\infty$.
If E contains an order linearly isometric copy of l_∞, then E_φ contains also such a copy.

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- Going to the second problem, that is, when the implication is true: (2) if $E_\varphi \in (P)$, then $E \in (P)$?

- We say that an Orlicz function φ satisfies the condition Δ_{2-str} **for all** $u \in \mathbb{R}_+$ ($\varphi \in \Delta_{2-str}(\mathbb{R}_+)$ for short) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that the inequality

$$\varphi((1+\delta)u) \leq (1+\varepsilon)\varphi(u) \tag{13}$$

holds for any $u \in \mathbb{R}_+$. We say that φ satisfies the condition Δ_{2-str} **for large [for small]**($\varphi \in \Delta_{2-str}(\infty)$ [$\varphi \in \Delta_{2-str}(0)$] for short) if for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $u_0 = u_0(\varepsilon) > 0$ such that $\varphi(u_0) < \infty$ [$\varphi(u_0) > 0$] and inequality (13) holds for any $u \geq u_0$ [$0 \leq u \leq u_0$], respectively ([CHKK, 2019]).

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- For any quasi-Banach ideal space E and any Orlicz function φ we say that φ satisfies the condition Δ^E_{2-str} ($\varphi \in \Delta^E_{2-str}$ for short) if:
  (1) $\varphi \in \Delta_{2-str}(\mathbb{R}_+)$ whenever neither $L_\infty \subset E$ nor $E \subset L_\infty$,
  (2) $\varphi \in \Delta_{2-str}(\infty)$ whenever $L_\infty \subset E$,
  (3) $\varphi \in \Delta_{2-str}(0)$ whenever $E \subset L_\infty$.

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- If $\varphi \in \Delta_{2-str}^{E}$ then $\varphi \in \Delta_2^E$ and the opposite implication is not true for non-convex function φ.

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- **Lemma.** Suppose one of the following three conditions holds:
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  (iii) $\varphi \in \Delta_{2-str}^{E}$, $1/a_E \leq \varphi(b_\varphi)$ and φ is strictly increasing on the interval $(0, \varphi^{-1}(1/a_E))$, where a_E is defined by formula (12), whenever $E \subset L_\infty$.
  Then for any $x \in E_\varphi$ such that $\|x\|_\varphi = 1$, we have $\rho_\varphi^E(x) = 1$.

- **Theorem.** Suppose one of the following three conditions holds:
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  If E_φ contains an order linearly isometric copy of l_∞, then E contains also such a copy.

Thank You very much for Your attention