Totally bounded sets in locally convex cones

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Cones

A **cone** is a set $\mathcal{P}$ endowed with an addition

$$(a, b) \to a + b$$

and a scalar multiplication

$$(\alpha, a) \to \alpha a$$

for $a, b \in \mathcal{P}$ and real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$.

For the scalar multiplication the usual associative and distributive properties hold, that is

$$\alpha(\beta a) = (\alpha\beta)a,$$

$$(\alpha + \beta)a = \alpha + \beta a$$

and

$$\alpha(a + b) = \alpha a + \alpha b$$

for all $a, b \in \mathcal{P}$ and $\alpha, \beta \geq 0$.
We have $1a = a$ and $0a = 0$ for all $a \in \mathcal{P}$.
The *cancelation law* , stating that

$$a + c = b + c \quad \text{implies} \quad a = b$$

however, is not required in general. It holds if and only if the cone $\mathcal{P}$ may be embedded into a real vector space.

Subcones

A subset $\mathcal{Q}$ of a cone $\mathcal{P}$ is called a **subcone** if

$$a + b \in \mathcal{Q} \quad \text{and} \quad \alpha a \in \mathcal{Q}$$

for all $a, b \in \mathcal{Q}$ and $\alpha \geq 0$.
We note that each subcone of $\mathcal{P}$ contains 0.

Examples

• Every vector space is a cone.
• The cones $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ and $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$, with the usual algebraic operations (especially $0 \cdot (+\infty) = 0$), are cones that are not embeddable in vector spaces.

preordered cone

A *preordered cone* (*ordered cone*) is a cone $\mathcal{P}$ endowed with a preorder (reflexive transitive relation) $\leq$ such that addition and multiplication by fixed scalars $r \in \mathbb{R}_+$ are order preserving, that is $x \leq y$ implies $x + z \leq y + z$ and $r \cdot x \leq r \cdot y$ for all $x, y, z \in \mathcal{P}$ and $r \in \mathbb{R}_+$.

(abstract) 0-neighborhood system

A subset $\mathcal{V}$ of the preordered cone $\mathcal{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

(i) $0 < v$ for all $v \in \mathcal{V}$;

(ii) for all $u, v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;

(iii) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

Neighborhoods

The elements v of $\mathcal{V}$ define upper, resp. lower, neighborhoods for the element a of $\mathcal{P}$ by

$$v(a) = \{b \in \mathcal{P} \mid b \leq a+v\}, \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} \mid a \leq b+v\},$$

creating the upper, resp. lower, topologies on $\mathcal{P}$. Their common refinement is called *symmetric* topology. We denote the neighborhoods of the symmetric topology as $v(a) \cap (a)v$ or $v^s(a)$ for $a \in \mathcal{P}$ and $v \in \mathcal{V}$.

For technical reasons we require that the elements of $\mathcal{P}$ to be *bounded below*, i.e. for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda > 0$. An element a of $(\mathcal{P}, \mathcal{V})$ is called bounded if it is also *upper bounded*, i.e. for every $v \in \mathcal{V}$ there is a $\lambda > 0$ such that $a \leq \lambda v$.

A **full locally convex cone** $(\mathcal{P}, \mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an abstract neighborhood system $\mathcal{V}$.

Finally, a **locally convex cone** $(\mathcal{P}, \mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$.

Example

The cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} : a \geq 0\}$ with (abstract) 0-neighborhood $\mathcal{V} = \{\varepsilon > 0 : \varepsilon \in \mathbb{R}\}$ are locally convex cones.

Example
Cone of convex sets.

Let $\mathcal{P}$ be a cone. A subset A of $\mathcal{P}$ is called **convex** if
$\alpha a + (1-\alpha)b \in A$, whenever $a, b \in A$ $0 \leq \alpha \leq 1$.
If we denote by $Conv(\mathcal{P})$ the set of all non-empty convex subsets of the cone $\mathcal{P}$, with the addition and scalar multiplication defined as:

$$A + B = \{a + b : \ a \in A \ , \ b \in B\}, \quad A, B \in Conv(\mathcal{P}) \ ,$$

$$\alpha A = \{\alpha a : \ a \in A\} \ , \ A \in Conv(\mathcal{P}), \quad \alpha \geq 0$$

$Conv(\mathcal{P})$ is again a cone.

We consider the order on $Conv(\mathcal{P})$ by

$$A \preceq B \quad if \quad A \subseteq \downarrow B,$$

where $\downarrow B = \{x \in \mathcal{P} | x \leq b \text{ for some } b \in B\}$ is the decreasing hull of the set B in $\mathcal{P}$. Note that $\downarrow B$ is again a convex subset of $\mathcal{P}$. The requirements for an ordered cone are easily checked.

The neighborhood system in $Conv(\mathcal{P})$ is $\overline{\mathcal{V}} := \{\overline{v} = \{v\} \mid v \in \mathcal{V}\}$, that is

$$A \preceq B + \overline{v} \quad if \quad A \subseteq \downarrow (B + \{v\})$$

for $A, B \in Conv(\mathcal{P})$ and $\overline{v} \in \overline{\mathcal{V}}$.
The cone $Conv(\mathcal{P})$ with (abstract) 0-neighborhood system $\overline{\mathcal{V}}$ is a locally convex cone.

Totally boundedness in locally convex cones

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A subset A of $\mathcal{P}$ is called **totally bounded** with respect to the symmetric topology if for every $v \in \mathcal{V}$, there is finite subset Φ of A such that

$$A \subseteq \bigcup_{x \in \Phi} v(x)v.$$

The totally boundedness of a set can be defined similarly under lower and upper topologies.

Theorem

For every subset A of the locally convex cone $(\mathcal{P}, \mathcal{V})$, the following are equivalent:

(i) A is totally bounded with respect to the symmetric topology.

(ii) For every $v \in \mathcal{V}$, there is finite subset Φ of A such that for every $a \in A$, one can find some $x \in \Phi$ such that $a \leq x + v$ and $x \leq a + v$.

(iii) For every $v \in \mathcal{V}$, there is finite subset Φ of A such that for each $a \in A$, one can find some $x \in \Phi$ such that $a \leq x + 2v$ and $x \leq a + 2v$.

(iv) For every $v \in \mathcal{V}$, there is totally bounded subset B such that for each $a \in A$, one can find some $b \in B$ such that $a \leq b + v$ and $b \leq a + v$.

Theorem

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. If $A \subseteq \mathcal{P}$ and $B \subseteq \mathcal{P}$ are totally bounded subsets with respect to the symmetric topology, then λA and $A + B$ are totally bounded with respect to the symmetric topology for all nonnegative real numbers λ.

Theorem

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. If $A \subseteq \mathcal{P}$ is totally bounded with respect to the symmetric topology, then $\overline{A}$ the closure of A so is.

Totally boundedness in locally convex Lattice cones

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a locally convex $\vee$- semilattice cone if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$, their supremum $a \vee b$ exists in $\mathcal{P}$ and if

($\vee$1) $(a + c) \vee (b + c) = a \vee b + c$ holds for all $a, b, c \in \mathcal{P}$,

($\vee$2) $a \leq c + v$ and $b \leq c + w$ for $a, b, c \in \mathcal{P}$ and $v, w \in \mathcal{V}$ implies that $a \vee b \leq c + (v + w)$.

Likewise, $(\mathcal{P},\mathcal{V})$ is a locally convex $\wedge$- semilattice cone if its order is antisymmetric and if for any two elements $a, b \in \mathcal{P}$, their infimum $a \wedge b$ exists in $\mathcal{P}$ and if

($\wedge$1) $\quad (a+c) \wedge (b+c) = a \wedge b + c$ holds for all $a, b, c \in \mathcal{P}$.

($\wedge$2) $\quad c \leq a + v$ and $c \leq b + w$ for $a, b, c \in \mathcal{P}$ and $v, w \in \mathcal{V}$ implies that $c \leq a \wedge b + (v + w)$.

If both sets of the above conditions i.e. ($\vee$1), ($\vee$2), ($\wedge$1) and ($\wedge$2)) hold, then $(\mathcal{P},\mathcal{V})$ is called a locally convex lattice cone.

If A and B are subsets of a locally convex $\vee$- semilattice cone, then we shall employ

$$A \vee B = \{a \vee b \ : \ a \in A \text{ and } b \in B\},$$

and in particular

$$A^{+} = A \vee \{0\} = \{a^{+} = a \vee 0 \ : \ a \in A\}.$$

For the subsets A and B of a locally convex $\wedge$- semilattice cone, we denote

$$A \wedge B = \{a \wedge b \ : \ a \in A \text{ and } b \in B\}.$$

Let $\mathcal{P}$ be a cone. A subset A of $\mathcal{P}$ is called **balanced** if $b \in A$ whenever $b = \lambda a$ or $b + \lambda a = 0$ for some $a \in A$ and $\lambda \in [0, 1]$. The **convex hull** coA is the smallest convex set that includes A. An easy argument shows that coA consists of all convex combinations of A. i.e.,

$$coA = \{\sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \quad \lambda_i \in [0,1] \quad \text{and} \quad \sum_{i=1} \lambda_i = 1\}.$$

Similarly, it can be seen that the set

$$cob(A) = \{\sum_{i=1}^{i=n} \lambda_i x_i : x_i \in A, \quad \lambda_i \in [0,1] \;\; \text{and} \;\; \sum_{i=1}^{i=n} \lambda_i \leq 1\}$$

is **the convex balanced hull** of A; i.e., the smallest convex and balanced set that includes A.

Theorem

(i) If A is a totally bounded subset of a locally convex $\vee$-semilattice cone $(\mathcal{P}, \mathcal{V})$ with respect to the upper topology, then $co(A)$ and $cob(A)$ are totally bounded with respect to the upper topology.

(ii) If A is a totally bounded subset of a locally convex $\wedge$-semilattice cone $(\mathcal{P}, \mathcal{V})$ with respect to lower topology, then $co(A)$ and $cob(A)$ are totally bounded with respect to the lower topology.

Theorem

(i) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\vee$-semilattice cone. If $A \subseteq \mathcal{P}$ and $B \subseteq \mathcal{P}$ are totally bounded subsets with respect to the upper topology, then $A \vee B$ is also totally bounded with respect to the upper topology.

(ii) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\wedge$-semilattice cone. if $A \subseteq \mathcal{P}$ and $B \subseteq \mathcal{P}$ are totally bounded subsets with respect to the lower topology, then $A \wedge B$ is also totally bounded with respect to the lower topology.

Corollary

(a) If $(\mathcal{P}, \mathcal{V})$ is a locally convex $\vee$ (or $\wedge$)-semilattice cone and $A \subseteq \mathcal{P}$ is a totally bounded subset with respect to the upper (or lower) topology, then A^+ is also totally bounded with respect to the upper (lower) topology.

(b) If $(\mathcal{P}, \mathcal{V})$ is a locally convex lattice cone and $A \subseteq \mathcal{P}$ is a totally bounded subset with respect to the symmetric topology then A^+ is also totally bounded with respect to the symmetric topology (and then with respect to the upper and lower topologies).

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Thank you for your attentions.

Questions?