An order theoretical analysis of atomic JBW-algebras

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Positivity XI, Ljubljana, 2023

Outline

$B(H)_{\mathrm{sa}}$ - set of all self-adjoint operators on a complex Hilbert space H with Jordan product given by

$$S \circ T = \tfrac{1}{2}(ST + TS). \tag{1}$$

$A := B(H)_{\mathrm{sa}}$ is a Jordan algebra, i.e. a commutative (not necessarily associative) algebra such that

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 \quad \text{for all } x, y \in A$$

and, moreover, a JB-algebra, i.e. a normed, complete Jordan algebra over $\mathbb{R}$ satisfying

$$\|x \circ y\| \leq \|x\|\|y\|, \quad \|x^2\| = \|x\|^2, \quad \|x^2\| \leq \|x^2 + y^2\|$$

for all $x, y \in A$.

- The identity operator I is an algebraic unit e in A.
- The spectrum $\sigma(x)$ of $x \in A$ is defined to be the set of $\lambda \in \mathbb{R}$ such that $x - \lambda e$ is not invertible in $\mathrm{JB}(x, e)$, the JB-subalgebra of A generated by x and e.
- The set A_+ of all elements in A that have a non-negative spectrum is a cone, its interior is the set of all elements with strictly positive spectrum. Note that $A_+ = \{x^2; x \in A\}$.
- The relation $\leq$ defined by $x \leq y$ whenever $y - x \in A_+$ has the properties
  (a) $x, y, z \in X$ and $x \leq y$ imply $x + z \leq y + z$,
  (b) $x \in X$, $0 \leq x$ and $\lambda \in [0, \infty)$ imply $0 \leq \lambda x$.

$A = B(H)_{\mathrm{sa}}$ is a partially ordered vector space.

- e is an order unit, i.e. for every $x \in A$ there is $\lambda \in \mathbb{R}$ such that $-\lambda e \leq x \leq \lambda e$.
- The norm in A is actually the order unit norm

$$\|x\| := \inf\{\lambda > 0; -\lambda e \leq x \leq \lambda e\}.$$

A is an order unit space (and, hence, a pre-Riesz space).

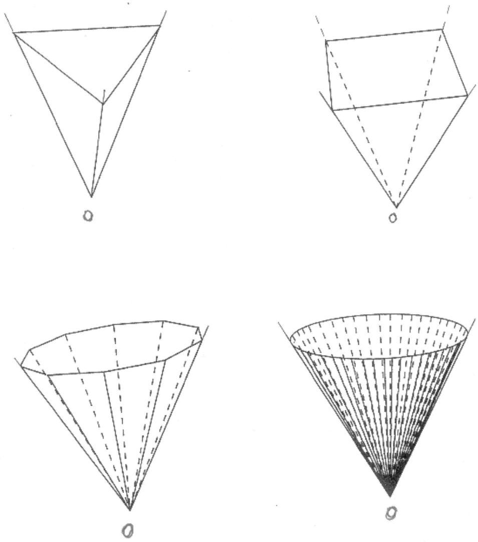
Definition

A partially ordered vector space (X, K) is an anti-lattice if the supremum of two elements exist only if they are comparable.

Theorem (Kadison 1951)

$B(H)_{\mathrm{sa}}$ is an anti-lattice.

Cones in $\mathbb{R}^3$: from lattice to anti-lattice



Disjointness

If (X, X_+) is a vector lattice, then $x, y \in X$ are called disjoint ($x \perp y$) if $|x| \wedge |y| = 0$, which is equivalent to

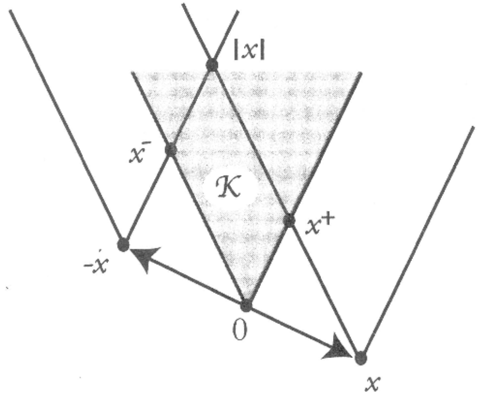
$$|x + y| = |x - y|.$$

If (X, X_+) is a partially ordered vector space, then $x, y \in X$ are called disjoint ($x \perp y$) if

$$\{x + y, -(x + y)\}^{\mathrm{u}} = \{x - y, -(x - y)\}^{\mathrm{u}},$$

where M^{u} denotes the set of upper bounds of $M \subseteq X$.

Idea: replace modulus $|x|$ by $\{x, -x\}^{\mathrm{u}}$



Theorem (K., Lemmens, van Gaans, 2014)

A partially ordered vector space is an anti-lattice if and only if there are no non-trivial disjoint positive elements.

In the space $B(H)_{\mathrm{sa}}$, it turns out that there are even no disjoint elements at all. We call such a partially ordered vector space a disjointness free anti-lattice.

- $B(H)_{\mathrm{sa}}$ is a JBW-algebra, i.e. a JB-algebra that is a dual space.
- A minimal element in the set of all non-zero projections of a JBW-algebra is called an atom.
- A JBW-algebra in which every non-zero projection dominates an atom is called atomic.

$B(H)_{\mathrm{sa}}$ is an atomic JBW-algebra.

Examples of atomic JBW-algebras [book of Alfsen, Shultz, 2003]:

(i) the self-adjoint bounded operators $B(H)_{\mathrm{sa}}$ on a real or complex Hilbert space H of dimension $d \geq 3$, or $B(\mathcal{H}_q)$ where $\mathcal{H}_q$ is a quaternionic Hilbert space of dimension $d \geq 3$, endowed with the product (1),

(ii) the spin factors $H \oplus \mathbb{R}$, where H is a real Hilbert space of dimension at least 2, with the multiplication

$$(x, \lambda) \circ (y, \mu) := (\mu x + \lambda y, \langle x, y \rangle + \lambda\mu). \tag{2}$$

The cone C of squares in $H \oplus \mathbb{R}$ equals $C = \{(x, \lambda); \sqrt{\langle x, x \rangle} \leq \lambda\}$.

(iii) the 3×3 self-adjoint matrices $M_3(\mathbb{O})_{\mathrm{sa}}$ with entries from the octonions $\mathbb{O}$, endowed with the product (1).

Theorem

Every atomic JBW-algebra equals the algebraic direct sum of atomic JBW-algebras that are isomorphic as JBW-algebras to those listed in (i)–(iii).

Joint work with Mark Roelands and Onno van Gaans:

A order theoretical analysis of all the so-called 'factors' in (i)–(iii)

B characterization of disjointness, bands, projection bands in atomic JBW-algebras

  idea: investigate order direct sums of order unit spaces

C study of disjointness preserving operators

A We show that every of the factors (i)-(iii) is a disjointness free anti-lattice.

Strategy:

- Every factor is an order unit space.
- (Kadison, 1951): Every order unit space has a functional representation, i.e. there is a compact Hausdorff space Ω and a linear bipositive map $\Phi\colon X \to \mathrm{C}(\Omega)$.
- Disjointness in the order unit space is equivalent to disjointness in the functional representation.

Construction of the functional representation:

Let (X, K, u) be an order unit space equipped with the u-norm $\|\cdot\|_u$. X' denotes the (norm) dual space of X and $K' := \{\varphi \in X';\ \varphi[K] \subseteq [0, \infty)\}$ the dual cone. The set

$$\Sigma = \{\varphi \in K';\ \varphi(u) = 1\}$$

is a base of K' (i.e. Σ is convex and every $\psi \in K'$ has a unique representation $\psi = \lambda\varphi$ with $\varphi \in \Sigma$ and $\lambda \in [0, \infty)$).

- By the Banach-Alaoglu theorem, the closed unit ball B' of X' is weakly-$*$ compact.
- As Σ is a weakly-$*$ closed subset of B', Σ is weakly-$*$ compact in X', i.e. Σ equals the weak-$*$ closure of the convex hull of the extreme points of Σ by the Krein-Milman theorem.

Denote the set of all extreme points of Σ by

$$\Lambda := \text{ext}(\Sigma).$$

(In general, Λ need not be weakly-$*$ closed, not even if X is finite dimensional.)
Denote by $\overline{\Lambda}$ the weak-$*$ closure of Λ in Σ, hence $\overline{\Lambda}$ is a compact Hausdorff space. Define

$$\Phi\colon X \to \mathrm{C}(\overline{\Lambda}), \qquad x \mapsto (\varphi \mapsto \varphi(x)).$$

- Φ is linear and maps u to the constant-1 function
- Let $x \in X$. Then $x \in K$ if and only if for every $\varphi \in \Lambda$ one has $\varphi(x) \geq 0$.
  Consequently, Φ is bipositive.

A partially ordered vector space X is called a pre-Riesz space if there exist a Riesz space Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y, i.e.,
for every $y \in Y$ one has $y = \inf\{z \in i[X]; z \geq y\}$.

The pair (Y, i) is then called a vector lattice cover of X.

Proposition (van Gaans, K., 2006)

Let X be a pre-Riesz space with vector lattice cover (Y, i). Then one has for every $x, y \in X$

$$x \perp y \iff i(x) \perp i(y).$$

The functional representation of an order unit space yields a vector lattice cover:

Theorem (Lemmens, van Gaans, K., 2014)

If (X, K, u) is an order unit space u, then $\Phi[X]$ is order dense in $\mathrm{C}(\overline{\Lambda})$, i.e. $(\mathrm{C}(\overline{\Lambda}), \Phi)$ is a vector lattice cover of X.

Every order unit space is a pre-Riesz space.

Disjointness of x and y in X is equivalent to pointwise disjointness of $\Phi(x)$ and $\Phi(y)$ in $\mathrm{C}(\overline{\Lambda})$.

Theorem (Roelands, K., van Gaans, 2023)

The factors (i)-(iii) in the factor decomposition of an atomic JBW-algebras are all disjointness free anti-lattices.

Let (X, X_+) be a partially ordered vector space.
The following *order theoretical notions* are of interest:

- For $M \subseteq X$ define the disjoint complement as $M^{\mathrm{d}} := \{x \in X; \forall m \in M \colon x \perp m\}$.
- $M \subseteq X$ is called a band, if $M = M^{\mathrm{dd}}$.
- A projection $P \colon X \to X$ is called an band projection if the range and kernel of P are bands.
  The range of a band projection is called a projection band.

Proposition (Glück, 2021)

Let X be a pre-Riesz space and $P \colon X \to X$ a linear operator. Then the following are equivalent:

(i) *P is a projection with $0 \leq P \leq I$.*

(ii) *There is a band B in X with $X = B \oplus B^d$, and P is the band projection onto B.*

Every projection band is directed.

B Order theoretical notions in direct sums of disjointness free anti-lattices

Proposition

Let (X_1, K_1, u_1) and (X_2, K_2, u_2) be order unit spaces. Then we have:

(a) $(X_1 \times X_2, K_1 \times K_2, (u_1, u_2))$ *is an order unit space.*

(b) *The functional representation* $\left(\mathrm{C}\left(\overline{\Lambda}_{X_1 \times X_2}\right), \Phi_{X_1 \times X_2}\right)$ *of* $X_1 \times X_2$ *satisfies*

$$\begin{aligned} \mathrm{C}\left(\overline{\Lambda}_{X_1 \times X_2}\right) &= \mathrm{C}\left(\overline{\Lambda}_{X_1}\right) \oplus \mathrm{C}\left(\overline{\Lambda}_{X_2}\right), \\ \Phi_{X_1 \times X_2}(x_1, x_2) &= \left(\Phi_{X_1}(x_1), \Phi_{X_2}(x_2)\right) \end{aligned}$$

for all $x_1 \in X_1$ *and* $x_2 \in X_2$.

(c) $X_1 \times \{0\}$ *and* $\{0\} \times X_2$ *are projection bands in* $X_1 \times X_2$.

Let $\mathcal{I}$ be a non-empty set and let $((V_i, C_i, u_i))_{i\in\mathcal{I}}$ be a collection of order unit spaces.

We define the order direct sum to be the vector space

$$\bigoplus_{i\in\mathcal{I}} V_i := \left\{ i \mapsto v_i : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} V_i;\ v_i \in V_i \text{ for every } i \in \mathcal{I} \text{ and } \sup_{i\in\mathcal{I}} \|v_i\|_{u_i} < \infty \right\} \tag{3}$$

with the cone $\{v \in \bigoplus_{i\in\mathcal{I}} V_i;\ v(i) \in C_i \text{ for every } i \in \mathcal{I}\}$.

Then $\bigoplus_{i\in\mathcal{I}} V_i$ is an order unit space with order unit $i \mapsto u_i$, which we denote by u.

Note that for every $v \in \bigoplus_{i\in\mathcal{I}} V_i$ we have that

$$\|v\|_u = \sup_{i\in\mathcal{I}} \|v(i)\|_{u_i}. \tag{4}$$

Let $((V_i, C_i, u_i))_{i\in\mathcal{I}}$ be a collection of JBW-algebras.

The algebraic direct sum of $(V_i)_{i\in\mathcal{I}}$ is the vector space given by (3) endowed with the norm given by (4) and componentwise multiplication.

$\bigoplus_{i\in\mathcal{I}} V_i$ is then a JBW-algebra.

If the V_i are atomic, then so is $\bigoplus_{i\in\mathcal{I}} V_i$.

The algebraic direct sum and the order direct sum of JBW-algebras coincide.

Disjointness in order direct sums of order unit spaces:

Lemma

Let $((V_i, C_i, u_i))_{i\in\mathcal{I}}$ *be a collection of order unit spaces with order direct sum* (V, C, u)*. Let* $v, w \in V$*. Then* v *and* w *are disjoint in* V *if and only if for every* $i \in \mathcal{I}$ *the elements* $v(i)$ *and* $w(i)$ *are disjoint in* V_i*.*

One has to show:

The components of an order direct sum of order unit spaces are projection bands that are pairwise disjoint.

Theorem (Roelands, K., van Gaans)

Let $M = \bigoplus_{i \in \mathcal{I}} M_i$ be an atomic JBW-algebra with its factor decomposition.

(i) *$B \subseteq M$ is a band if and only if $B = \bigoplus_{j \in \mathcal{J}} M_j$ for $\mathcal{J} \subseteq \mathcal{I}$, where it is understood that $B = \{0\}$ for $\mathcal{J} = \emptyset$.*

(ii) *Two non-zero $x, y \in M$ are disjoint if and only if there is a $\mathcal{J} \subseteq \mathcal{I}$ with $\mathcal{J} \neq \emptyset$ and $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$ such that $x \in \bigoplus_{i \in \mathcal{J}} M_i$ and $y \in \bigoplus_{i \in \mathcal{I} \setminus \mathcal{J}} M_i$.*

If X is a povs, a linear operator $T\colon X \to X$ is called disjointness preserving if for every $x, y \in X$ from $x \perp y$ it follows that $Tx \perp Ty$.

- C Under which conditions is the inverse of a disjointness preserving bijection disjointness preserving?

Theorem (Huijsmans, de Pagter, 1994, Koldunov 1995)

If X is a Banach lattice and $T\colon X \to X$ is a disjointness preserving linear bijection, then T^{-1} is disjointness preserving.

Theorem (Lemmens, van Gaans, K., 2018)

Let (X, K) be a finite-dimensional povs with closed generating cone K. If $T\colon X \to X$ is a disjointness preserving linear bijection, then T^{-1} is disjointness preserving.

In an atomic JBW-algebra, the inverse of a disjointness preserving bijection is not disjointness preserving, in general.
[van Gaans, K., Roelands 2023]

To do: If $M = \bigoplus_{i \in \mathcal{I}} V_i$ is an atomic JBW-algebra with the corresponding factor decomposition, characterize the disjointness preserving bijections on M with disjointness preserving inverse.

If $((W_i, K_i, w_i))_{i\in\mathcal{I}}$ is another family of order unit spaces and for every $i \in \mathcal{I}$ we have a linear map $T_i\colon V_i \to W_i$ such that for every $v \in \bigoplus_{i\in\mathcal{I}} V_i$ the map $i \mapsto T_i v(i)$ from $\mathcal{I}$ to $\bigcup_{i\in\mathcal{I}} W_i$ belongs to $\bigoplus_{i\in\mathcal{I}} W_i$, then we denote this map by $\bigoplus_{i\in\mathcal{I}} T_i$.

Theorem (Roelands,K.,van Gaans)

Let $((V_i, C_i, u_i))_{i\in\mathcal{I}}$ be a collection of order unit spaces that are disjointness free anti-lattices with order direct sum (V, C, u). Let $T\colon V \to V$ be a disjointness preserving linear bijection. Then T^{-1} is disjointness preserving if and only if there is a bijection $\sigma\colon \mathcal{I} \to \mathcal{I}$ and there are linear bijections $T_i\colon V_i \to V_{\sigma(i)}$ such that $T = \bigoplus_{i\in\mathcal{I}} T_i$.

Corollary

Let $M = \bigoplus_{i\in\mathcal{I}} M_i$ *be an atomic JBW-algebra with the corresponding factor decomposition, and* $T\colon M \to M$ *be a disjointness preserving linear bijection. Then* T^{-1} *is disjointness preserving if and only if there is a bijection* $\sigma\colon \mathcal{I} \to \mathcal{I}$ *and there are linear bijections* $T_i\colon M_i \to M_{\sigma(i)}$ *such that* $T = \bigoplus_{i\in\mathcal{I}} T_i$*.*

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Thanks for your attention!