# The Gleason Metric and Fibers of $\mathcal{H}^\infty(B_{c_0})$

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POSITIVITY 2023

July 10-14, 2023

# Background

Given a complex Banach space $X$, we denote the algebra of all bounded holomorphic functions on the open unit ball $B_X$ of $X$ by $\mathcal{H}^\infty(B_X)$.

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It is well known that when endowed with the supremum norm, $\mathcal{H}^\infty(B_X)$ is a commutative Banach algebra with identity.

We also denote by $\mathcal{M}(\mathcal{H}^\infty(B_X))$ the spectrum (or maximal ideal space) of $\mathcal{H}^\infty(B_X)$ that consists of the set of all non-zero complex valued homomorphisms $\psi : \mathcal{H}^\infty(B_X) \mapsto \mathbb{C}$, endowed with the weak-star topology that makes it a compact Hausdorff space.

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The *Gleason metric* $\rho$ on the spectrum $\mathcal{M}(\mathcal{H}^\infty(B_X))$ is defined so that $\rho(\varphi, \psi)$ is the supremum of $|\psi(f)|$ over all $f \in \mathcal{H}^\infty(B_X)$ satisfying $\|f\|_\infty := \sup_{x \in B_X}\{|f(x)| : x \in B_X\} \leq 1$ and $\varphi(f) = 0$.

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Let us point out that the Gleason metric $G$ on $\mathcal{M}(\mathcal{H}^\infty(B_X))$ is sometimes defined by the restriction of the dual norm on $\mathcal{H}^\infty(B_X)^*$ to $\mathcal{M}(\mathcal{H}^\infty(B_X))$, i.e., $G(\varphi, \psi) = \|\varphi - \psi\|^* = \sup\{|\varphi(f) - \psi(f)| \,:\, \|f\|_\infty \leq 1\}$. We prefer to use the metric $\rho$, because it allows us to obtain its explicit description.

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Both metrics $\rho$ and $G$ are related by the equation $\rho(\varphi,\psi) = \frac{4G(\varphi,\psi)}{4+G(\varphi,\psi)^2}$

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Since $X^*$, the dual space of $X$, is a subspace of $\mathcal{H}^\infty(B_X)$, we can define a natural surjective mapping $\pi : \mathcal{M}(\mathcal{H}^\infty(B_X)) \mapsto \overline{B_{X^{**}}}$ given by $\pi(\psi) := \psi|_{X^*}$, i.e. the restriction of $\psi$ to $X^*$

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For $z \in \overline{B_{X^{**}}}$, the *fiber* of the spectrum of $\mathcal{H}^\infty(B_X)$ at the point $z$ is defined by $\pi^{-1}(z) = \{\psi \in \mathcal{M}(\mathcal{H}^\infty(B_X)) : \pi(\psi) = z\}$. As usual, we denote the fiber $\pi^{-1}(z)$ by $\mathcal{M}_z(\mathcal{H}^\infty(B_X))$. By Goldstine's theorem, $\pi(\mathcal{M}(\mathcal{H}^\infty(B_X))) = \overline{B_{X^{**}}}$, so every fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_X))$ is nonempty.

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If $z \in B_{X^{**}}$, then we define $\tilde{\delta}_z(f) := AB(f)(z)$, where $AB(f)$ denotes the Aron-Berner extension of $f$. Since $\tilde{\delta}_z \in \mathcal{M}_z(\mathcal{H}^\infty(B_X))$, we can consider $B_{X^{**}}$ to be contained in $\mathcal{M}(\mathcal{H}^\infty(B_X))$. Canonically, we will identify $z$ with $\tilde{\delta}_z$.

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We are interested in studying algebras of holomorphic functions. We say that a vector-valued function $\phi : B_X \mapsto Z$ is holomorphic, if $\phi$ is Fréchet differentiable. In particular, if $Z = Y^*$ for a Banach space $Y$, then this is equivalent to say that $\phi$ is $w^*$-holomorphic, that is, $y \circ \phi : B_X \mapsto \mathbb{C}$ is holomorphic for all $y \in Y$.

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We concentrate on the algebra $\mathcal{H}^\infty(B_X)$, $X = c_0$, and our main interest is the structure of the fibres $\pi^{-1}(z)$ for $z \in \overline{B_{\ell_\infty}}$.

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**Theorem (Cole, Gamelin and Johnson, Michigan Math. J., 1992)**

*For every point $z \in B_{\ell_\infty}$ there exists an analytic injection of the ball $B_{\ell_\infty}$ into $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$, which is also an isometry from the Gleason metric $G$ of $B_{\ell_\infty}$ to the Gleason metric $G$ of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.*

# Main Theorem

**Theorem (Aron, Falcó, García, and Maestre, Studia Math., 2018)**

*For every distinguished boundary point $z$ of $\overline{B_{\ell_\infty}}$ there exists an analytic injection of the ball $B_{\ell_\infty}$ into $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$, which is also a homeomorphism on its image.*

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**Theorem (Main Theorem, CFGJM)**

*For every $z \in \overline{B_{\ell_\infty}}$ there is an analytic injection of the ball $B_{\ell_\infty}$ into the fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ which is an isometry from the Gleason metric of $B_{\ell_\infty}$ to the Gleason metric of $\mathcal{M}(H^\infty(B_{c_0}))$.*

# The Key Case

**Theorem (CFGJM)**

*If $z = (z_n) \in \overline{B_{\ell_\infty}}$ and there exists a subsequence $(z_{j_n})_{n\in\mathbb{N}}$ such that $|z_{j_n}| < 1$ for every $n \in \mathbb{N}$ and $(|z_{j_n}|)_{n\in\mathbb{N}}$ is a strictly increasing sequence converging to 1, then there exists an analytic injection of the ball $B_{\ell_\infty}$ into the fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$, which is an isometry for the Gleason metric of $B_{\ell_\infty}$ to the Gleason metric of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.*

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Sketch of Proof :

We denote by $\mathbb{N}_1 = \{j_n : n \in \mathbb{N}\}$ and $\mathbb{N}_2 = \mathbb{N} \setminus \mathbb{N}_1 = \{i_l : l = 1, \ldots, N\}$, where $N$ is the cardinality of the set $N_2$, possibly 0 or $\infty$

# Sketch of Proof

We write each element $\lambda \in \ell_\infty$ as $(\mu, \eta) \in \ell_\infty \times \ell_\infty^N$, where $\mu_n = \lambda_{j_n} \quad (n \in \mathbb{N})$ and $\eta_l = \lambda_{i_l} \quad (l = 1, \ldots, N)$. Following this notation, we write $z = (u, v)$.

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For each $k, m \in \mathbb{N}$ choose an integer $m_k$ and $0 < r_m < 1$ so that $m_{k+1} > m_k + k$ and $\{r_m\}$ increases strictly to 1. Choose $0 < \alpha_k < 1$ so that $\{\alpha_k\}$ increases strictly to 1 very rapidly and

$$1 - \alpha_k \leq \frac{1}{2^k}(1 - |u_{m_{k+1}}|) \tag{1}$$

for every $k \in \mathbb{N}$.

# Sketch of Proof

We define the holomorphic mapping

$$\Psi_k^m : B_{\ell_\infty} \to B_{\ell_\infty}$$

by

$$\Psi_k^m(\lambda) = (\Phi_k^m(\lambda), r_m v)$$

for every $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in B_{\ell_\infty}$, where

$\Phi_k^m : B_{\ell_\infty} \to B_{c_0}$ is the mapping defined by

$$\Phi_k^m(\lambda) = \left( r_m u_1, \ldots, r_m u_{m_k}, \frac{\alpha_k - \lambda_1}{1 - \alpha_k \lambda_1}, \ldots, \frac{\alpha_k - \lambda_k}{1 - \alpha_k \lambda_k}, 0, 0, \ldots \right)$$

for all $\lambda = (\lambda_n)_{n=1}^{\infty} \in B_{\ell_\infty}$.

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$\Phi_k^m$ is weakly holomorphic; hence $\Phi_k^m$ is also holomorphic.

Given $\eta \in B_{\ell_\infty}$ we identify $\eta$ with the point evaluation homomorphism $\tilde{\delta}_\eta \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$'

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We have

$$\tilde{\Phi}_k^m, \tilde{\Psi}_k^m : B_{\ell_\infty} \mapsto \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \subset \mathcal{H}^\infty(B_{c_0})^*$$

defined as $\tilde{\Phi}_k^m(\eta)(f) = f(\Phi_k^m(\eta))$ and $\tilde{\Psi}_k^m(\eta)(f) = AB(f)(\Psi_k^m(\eta))$ respectively.

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To simplify the notation we will identify $\tilde{\Phi}_k^m$ and $\tilde{\Psi}_k^m$ with $\Phi_k^m$ and $\Psi_k^m$, respectively.

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Now, the mappings $\Phi_k^m$ and $\Psi_k^m$ can be now considered as holomorphic mappings from $B_{\ell_\infty}$ to $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.

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By Ascoli theorem, there exist holomorphic mappings $\Phi^m, \Psi^m : B_{\ell_\infty} \to \mathcal{M}(H^\infty(B_{c_0}))$, which are accumulation points of $(\Psi_k^m)_{k\in\mathbb{N}}$ and $(\Psi_k^m)_{k\in\mathbb{N}}$, respectively.

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We can check that

$$\Psi^m(\lambda) = \big(\Phi^m(\lambda), r_m v\big)$$

and

$$\Psi^m(\lambda)(f) = \Phi^m(\lambda)\left(\mu \in B_{\ell_\infty} \mapsto AB(f)\left(\mu, r_m v\right)\right),$$

for every $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in B_{\ell_\infty}$ and every $f \in \mathcal{H}^\infty(B_{c_0})$.

# Sketch of Proof

By Ascoli theorem again, we can find holomorphic mappings $\Phi, \Psi : B_{\ell_\infty} \to \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, which are accumulation points of $\{\Phi^m\}_{m\in\mathbb{N}}$ and $\{\Psi^m\}_{m\in\mathbb{N}}$ respectively in the compact-open topology.

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We can check that $\Psi(\lambda) \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ for every $\lambda \in B_{\ell_\infty}$.

We can show that $\Psi$ is an analytic injection of the ball $B_{\ell_\infty}$ into $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, which is an isometry from the Gleason metric of $B_{\ell_\infty}$ to the Gleason metric of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.

**Theorem**

*Suppose that $X$ is a Banach space with a normalized shrinking basis $\{e_j\}_{j\in\mathbb{N}}$ and there exists a positive integer $N$ satisfying*

$$\sum_{j=1}^{\infty} |e_j^*(x)|^N < \infty$$

*for all $x = \sum_{j=1}^{\infty} e_j^*(x)e_j$ in $X$. Then for any $z \in B_X$, there exists an analytic injection of the ball $B_{\ell_\infty}$ into the fiber $\mathcal{M}_z(\mathcal{A}_u(B_X))$.*

# Generalization

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THANK YOU FOR YOUR ATTENTION!