Diagonal processes

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- We provide some applications

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- We formulate a concrete statement for it.
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    1. a constructive countable version;
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    1. New short proofs
    2. Answers to some open questions

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Theorem

Let (x_n) be a sequence in (a nonempty set) E and $(\mathcal{P}_k)_{k\in\mathbb{N}}$ a sequence of properties.

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2. *for each k and for each subsequence (y_n) of (x_n) there is a further subnet (z_n) satisfying $\mathcal{P}_k$.*

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   Then *there is a subsequence (z_θ) of (x_α) satisfying all properties $\mathcal{P}_k$.*

Some definitions : Nets and subnets

Definition

A net in a set S is a map

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- We just write $(x_\alpha)_{\alpha \in A}$.
- A is directed means that A is equipped with a binary relation (preorder) $\leq$ such that
  1. $x \leq x$
  2. $x \leq y$ and $y \leq z \implies x \leq z$.
  3. $\forall \alpha, \beta \in A \Longrightarrow \exists\ \gamma : \gamma \geq \alpha$ and $\gamma \geq \beta$.

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Then *there is a subnet* (z_θ) *of* (x_α) *satisfying* $\mathcal{P}_k$ *for all* k.

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Denote by $x = x^0 = (x_\alpha)_{\alpha \in A}$ the initial net.

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- For some cofinal maps $\varphi_n : A_n \longrightarrow A_{n-1}$, $x^n_{\alpha_n} = x^{n-1}_{\varphi_n(\alpha_n)}$.

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- For some cofinal maps $\varphi_n : A_n \longrightarrow A_{n-1}$, $x^n_{\alpha_n} = x^{n-1}_{\varphi_n(\alpha_n)}$.
- The subnet x^n satisfies $\mathcal{P}_k$ for $1 \leq k \leq n$.

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- Let B be the set of all β of the form

$$\beta = (\beta_0, \beta_1, ..., \beta_k) \in A_0 \times A_1 \times ... \times A_k,$$

where $k \in \mathbb{N}$; $\qquad \beta_{i-1} = \varphi_i(\beta_i), \qquad 1 \leq i \leq k.$

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- B is ordered as follows :

$$\left(\beta_0, \beta_1, ..., \beta_p\right) \preccurlyeq (\gamma_0, \gamma_1, ..., \gamma_q) \Leftrightarrow p \le q \text{ and } \beta_i \le \gamma_i \text{ for } 0 \le i \le p.$$

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1. $(B, \preccurlyeq)$ is directed.
2. $y = \left(y_\beta\right)_{\beta \in B}$ is a subnet of (x_α) via the cofinal map

$$\varphi : B \longrightarrow A; \qquad \left(\beta_0, \beta_1, ..., \beta_p\right) \longmapsto \beta_0.$$

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- To conclude see that $\qquad y$ is an eventual subnet of x^n , $\forall n$.

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Theorem (Banach-Alaoglu Theorem, separable case)

If a Banach space X is separable then the unit ball B_{X^*} of X^* is weakly compact.*

Applications : un compact operators

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Theorem

Let E be a Banach space and X a Banach lattice. Then the set $K_{un}(E, X)$ of all un-compact operators from E to X is a closed subspace of $L(E, X)$.

Diagonal process : The general case

Theorem

Let $(x_\alpha)_{a \in A}$ *be a net in some space* X *and consider a family of properties* $(\mathcal{P}_i)$. **W'e assume that**

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   Then *there is a subnet* (u_β) *of* (x_α) *satifying all properties* $\mathcal{P}_i$.

Applications : Short and neat proofs

We present here very short proofs of some classical results.

Theorem (Tychonoff)

The product of compact spaces is compact.

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Proof.

Let $X = \prod_{i \in I} X_i$, X_i compact and (f_α) be a net in X.

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- By the Diagonal Process (f_α) has a subnet (g_β) such that $g_\beta(i)$ converges for all $i \in I$.

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- We are done.

$\square$

Applications : Short and neat proofs

Theorem (Banach-Alaoglu)

Let X be a Banach space. Then the unit ball B_{X^*} is weak compact.*

Proof.

Let (f_α) be a net in B_{X^*}.

1. Then for every $x \in E$, every subnet of (f_α) has a subnet g_β such that $g_\beta(x)$ converges.
2. It follows that (f_α) has a subnet g_β which converges pointwise on E.
3. Its pointwise limit is linear and $|g(x)| \leq \|x\|$. So $g \in B_{X^*}$ and we are done.

□

Theorem (Ascoli)

Let A be a subset of $C(X)$, X compact. Assume that A is equicontinuous, bounded and closed. Then A is compact.

Applications : New proofs

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Let X be a vector lattice

1. $x_\alpha \xrightarrow{o_1} x \Longrightarrow x_\alpha \xrightarrow{o} x$.
2. $x_\alpha \xrightarrow{o} x$ *in* $X \Longleftrightarrow x_\alpha \xrightarrow{o_1} x$ *in* X^δ.
3. $\xrightarrow{o} \Longleftrightarrow \xrightarrow{o_1}$ *if X is dedekind complete.*

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What's new?

Theorem

Let (x_α) is a net in a vector lattice X.

1. *If* $x_\alpha \xrightarrow{o} x$ *then* $y_\gamma \xrightarrow{o_1} x$ *for some subnet* (y_γ) *of* (x_α).
2. *If* $x_\alpha \xrightarrow{uo} x$ *then* $y_\gamma \xrightarrow{uo_1} x$ *for some subnet* (y_γ) *of* (x_α).

An other application

The following answers a question of Taylor.

Theorem

Let (X, τ) *be a Hausdorff locally solid vector lattice.*

1. τ *has the o-Lebesgue property iff it has the* o_1*-Lebesgue property.*
2. τ *has the uo-Lebesgue property iff it has the* uo_1*-Lebesgue property.*

Thank you

for your attention