Free objects in analytic categories

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Context and Outline

The work presented in this talk builds upon the work done in:
M. de Jeu. *Free Vector Lattices and Free Vector Lattice Algebras.*

Outline

(I) Examples of free objects.

(II) Uniform approach for constructing normed free objects.

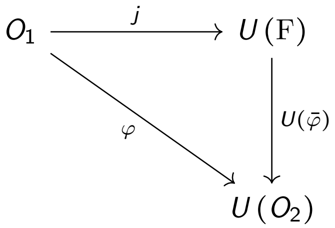
(III) Inverse limit construction for locally convex free objects.

Definition of a free object

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories with $U : \mathbf{C}_2 \to \mathbf{C}_1$ a 'faithful' functor.

Fix $O_1 \in \mathbf{C}_1$. A pair (F, j) with $\mathrm{F} \in \mathbf{C}_2$ and $j : O_1 \to U(\mathrm{F})$ a morphism in $\mathbf{C}_1$ is called **a free object over O_1 of $\mathbf{C}_2$ with respect to U** if the following universal property holds:

For every $O_2 \in \mathbf{C}_2$ and every morphism $\varphi : O_1 \to U(O_2)$ in $\mathbf{C}_1$, there exists a unique morphism $\bar{\varphi} : \mathrm{F} \to O_2$ in $\mathbf{C}_2$ such that the following diagram commutes in $\mathbf{C}_1$.



Definition of a free object

Remarks

(i) Free objects (F, j) are unique up to a unique isomorphism in $\mathbf{C}_2$, when they exist.

(ii) For categories $\mathbf{C}_1$ and $\mathbf{C}_2$, we write $\left(\mathrm{F}_{\mathbf{C}_1}^{\mathbf{C}_2}(O_1), j\right)$ for the free object (F, j) over O_1 of $\mathbf{C}_2$ when the faithful functor $U : \mathbf{C}_2 \to \mathbf{C}_1$ is understood.

(iii) Examples of faithful functors:
- The 'forgetful functor' $U : \mathbf{Ban} \to \mathbf{Set}$, which sends a Banach space X to the underlying set of X.
- Let $M \in \mathbb{R}_+ \backslash \{0\}$. The 'ball functor' $B_M : \mathbf{Ban} \to \mathbf{Set}$, which sends a Banach space X to the underlying set of the closed unit ball centred at the origin with radius M, denoted as $\mathbf{B}_M(X)$.

Categories under consideration

Algebraic categories:

	Objects	**Morphisms**
Set	Sets	Functions
VS	Vector spaces	Linear maps
Alg	Algebras	Multiplicative linear maps
VL	Vector lattices	Lattice homomorphisms
VLA	Vector lattice algebras	Multiplicative lattice homomorphisms

Categories of normed objects:

	Objects	**Morphisms**
Ban	Banach spaces	Contractive linear maps
BA	Banach algebras	Contractive algebra homomorphisms
BL	Banach lattices	Contractive lattice homomorphisms
BLA	Banach lattice algebras	Contractive VLA homomorphisms

Examples of algebraic free objects

(1) **Vector spaces:** Let S be a non-empty set.

- Let V_S denote the collection of functions $f : S \to \mathbb{R}$ with finite support. Define $j : S \to V_S$ where $s \mapsto \delta_s$. Then (V_S, j) is the free vector space over S (with respect to the forgetful functor) and $j[S]$ is a basis for V_S.

(2) **Algebras:** Let S be a non-empty set.

- Consider the (non-commutative) polynomial ring $\mathbb{K}[S]$ with indeterminates $\{X_s : s \in S\}$ along with the map $j : S \to \mathbb{K}[S]$ where $s \mapsto X_s$. The pair $(\mathbb{K}[S], j)$ is the free algebra over the set S.
- We can generate $\mathbb{K}[S]$ by equipping the free vector space V_S with a discrete convolution.

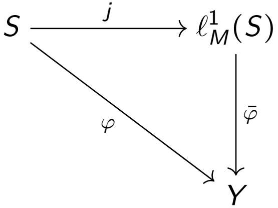
(3) **In general:** Universal algebra theory tells us that free objects exist in all 'equational' algebraic categories. Existence is guaranteed when no concrete model can be found.

Example: Vector lattice algebras.

Construction of an analytic free object

Let S be a non-empty set and fix $M \in \mathbb{R}_+ \backslash \{0\}$.

Consider the free vector space (V_S, j) over S. Equip V_S with the ℓ^1-norm weighted by the constant M. The completion of $\left(V_S, \|\cdot\|_{1,M}\right)$ is $\ell^1_M(S)$ and the pair $\left(\ell^1_M(S), j\right)$ is the **free Banach space over** S (with respect to the ball functor B_M):



If we omit the bound M above, in the case of $|S| = \infty$, we can construct maps $\varphi : S \to X$ which 'grow too quickly' to be factored as a bounded morphism. As a result, there is no free Banach space over an infinite set with respect to the forgetful functor.

Construction of free objects

Remarks

(i) For a non-empty set S, $\left(\mathrm{F}^{\mathbf{BL}}_{\mathbf{Set}}(S), j\right)$ is constructed by equipping $\left(\mathrm{F}^{\mathbf{VL}}_{\mathbf{Set}}(S), j_0\right)$ with an appropriate norm in [1]. Note that the parameter $M = 1$ is implicitly used in the construction.

(ii) Analytic free objects can be constructed from algebraic free objects in many different contexts by means of a uniform approach which is reminiscent of the above example.

(iii) This approach only requires existence of the algebraic free object. No concrete model is needed.

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[1] B. de Pagter and A.W. Wickstead. *Free and projective Banach lattices.*

Uniform approach

Recipe

(1) Start with the existence of an algebraic free object (F, j).
(Fully automated by the universal algebra theory.)

(2) Equip F with a seminorm ρ defined using the universal property of (F, j).

(3) Quotient out kernel of ρ and complete, if necessary.

This recipe is already known:

- Similar approach used in [2].
- Present version first recorded in [3].

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[2] N.C. Phillips. *Inverse limits of C^*-algebras.*

[3] M. de Jeu, M. A. Taylor, V. G. Troitsky. *The Wickstead problems on Banach lattice algebras.* (Unpublished working document)

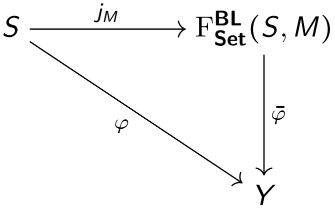
Example of construction

Example

Let S be a non-empty set and fix $M \in \mathbb{R}_+ \setminus \{0\}$.

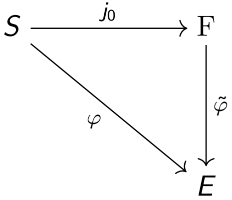
We construct $\left(\mathrm{F}^{\mathbf{BL}}_{\mathbf{Set}}(S, M), j_M\right)$, which satisfies the following universal property:

For every Banach lattice Y and every set map $\varphi : S \to \mathbf{B}_M(Y)$ there exists a unique $\bar{\varphi} : \mathrm{F}^{\mathbf{BL}}_{\mathbf{Set}}(S, M) \to Y$ in $\mathbf{BL}$ such that



Example of construction

Step 1: We start with the free vector lattice (F, j_0), which satisfies the following universal property: For every vector lattice E and every set map $\varphi : S \to E$ there exists a unique lattice homomorphism $\tilde{\varphi} : \mathrm{F} \to E$ such that



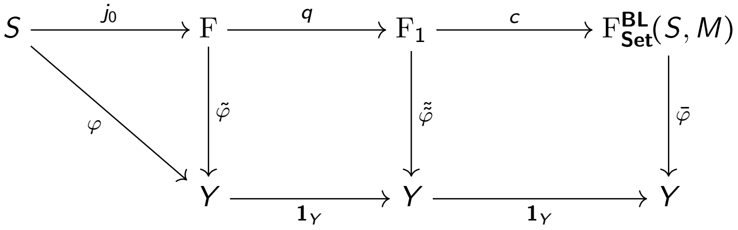
Step 2: Define $\rho : \mathrm{F} \to [0, \infty]$ where

$$\rho(x) := \sup\left\{ \|\tilde{\varphi}(x)\|_Y \;\middle|\; \begin{array}{l} Y \text{ Banach lattice,} \\ \tilde{\varphi} \text{ unique factorisation of } \varphi : S \to \mathbf{B}_M(Y) \text{ via } (\mathrm{F}, j_0) \end{array} \right\}$$

Example of construction

For a set map $\varphi : S \to \mathbf{B}_M(Y)$, by the universal property of (F, j_0) we have $\|\tilde{\varphi}(j_0(s))\|_Y = \|\varphi(s)\|_Y \leq M$. Thus ρ is finite on $j_0[S]$. Since the subset $j_0[S]$ generates F as a vector lattice, we conclude that ρ is finite on F.

Step 3: The quotient $\mathrm{F}_1 \coloneqq \mathrm{F}/\ker\rho$ is a normed vector lattice and the completion $\hat{\mathrm{F}}_1$ is a Banach lattice, which we denote as $\mathrm{F}^{\mathbf{BL}}_{\mathbf{Set}}(S, M)$.



Define $j_M \coloneqq c \circ q \circ j_0$. The pair $\left(\mathrm{F}^{\mathbf{BL}}_{\mathbf{Set}}(S, M), j_M\right)$ satisfies the desired universal property.

Example of construction

Remarks

(i) In conclusion, the uniform recipe described above delivers the existence of many analytic free objects. (143 known.)

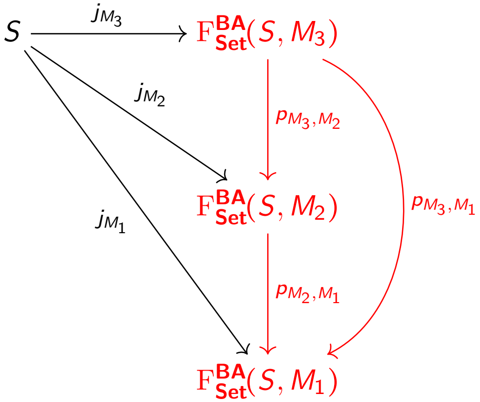
(ii) These include analytic free objects over algebraic structures, e.g. $\left(\mathrm{F}^{\textbf{BLA}}_{\textbf{VL}}(S, M), j_M\right)$, as well as analytic free objects over 'weaker' analytic structures, e.g. $\left(\mathrm{F}^{\textbf{BLA}}_{\textbf{Ban}}(S, M), j_M\right)$.

(iii) This method can be extended to prove the existence of locally convex free objects by means of an inverse limit construction.

(iv) We outline this construction as it relates to Banach algebras.

Inverse limit construction

Let S be a non-empty set and take $M_3 \geq M_2 \geq M_1 \geq 0$. Consider:



$\mathcal{I} \coloneqq \Big(\big(\mathrm{F}^{\mathbf{BA}}_{\mathbf{Set}}(S, M)\big)_{M \geq 0}, \big(p_{M_2,M_1}\big)_{M_2 \geq M_1} \Big)$ forms an inverse system in **BA**.
Also, $(S, (j_M)_{M \geq 0})$ forms a compatible system over $\mathcal{I}$.

Inverse limit construction

- The inverse system of free objects $\mathcal{I}$ has an inverse limit $(\mathcal{F}, (p_M)_{M\geq 0})$ in the **category of inverse limits of Banach algebras**, call it **X**.
- Since $(S, (j_M)_{M\geq 0})$ is compatible over $\mathcal{I}$, there exists a map $j : S \to \mathcal{F}$.
- By means of an **unmentioned general categorical lemma**, we know that the pair $(\mathcal{F}, j)$ is the free object over S in the category of inverse limits **X**.
- From [4, Chapter 3.3], we know that the inverse limits of Banach algebras are precisely the **complete locally m-convex algebras**.
- As a result, the pair $(\mathcal{F}, j)$ is the free complete locally m-convex algebra over S.

[4] A. Mallios. *Topological Algebras. Selected Topics.* (1986)

Inverse limit construction

Definition

A locally m-convex algebra is an algebra equipped with a topology generated by a separating family of submultiplicative seminorms.

This inverse limit construction also gives us another 24 locally convex free objects over algebraic objects.

Example

Let $S = \{s\}$. The free complete locally m-convex (complex) algebra over a point is the pair $((H(\mathbb{C}), \tau), j)$.

$(H(\mathbb{C}), \tau)$ is the algebra of entire functions over $\mathbb{C}$ equipped with the topology of uniform convergence on compact sets!

Thank you for your attention!